Implementing full MHD equations in nonlinear code JOREK

E.V. van der Plas

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1 Solving the MHD equations

The MHD code `JOREK` aims to solve the full MHD equations using a fully implicit scheme.

2 The MHD equations

The formulation of the resistive MHD equations in `JOREK` is as follows:

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (D \nabla \rho) + S_\rho, \tag{1}$$
$$\rho \partial_t \mathbf{v} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla(\rho T) + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \tag{2}$$
$$\partial_t T = -\mathbf{v} \cdot \nabla T + (\gamma - 1) T \, \nabla \cdot \mathbf{v} + \nabla \cdot (\bar{\mathsf{K}} \nabla T) + S_T, \tag{3}$$
$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}). \tag{4}$$

The apparent absence of magnetic monopoles implies $\nabla \cdot \mathbf{B} = 0$, so that we can write

$$\mathbf{B} = \nabla \times (\mathbf{A} + \nabla \Phi),$$

where Φ is a gauge function, i.e. without physical consequence. When we use the gauge $\Phi = 0$, the induction equation (4) becomes

$$\partial_t \mathbf{A} = \mathbf{v} \times (\nabla \times \mathbf{A}) - \eta (\nabla \times \nabla \times \mathbf{A}). \tag{5}$$

2.1 Toroidal coordinate system

As many fusion devices these days have a toroidal shape, it is instructive to adapt a toroidal coordinate system (R, Z, ϕ), so that we can write the (equilibrium) magnetic field as

$$\mathbf{B}_0 = \nabla \times \mathbf{A}_0 = F \nabla \phi + \nabla \psi_0 \times \nabla \phi, \tag{6}$$

where ϕ is the toroidal angle, and ψ_0 is the poloidal flux. This leads to the formulation of a co- and contravariant basis,

$$\begin{aligned} \mathbf{a}_1 &= J(\nabla Z \times \nabla \phi), & \mathbf{a}^1 &= \nabla R, \\ \mathbf{a}_2 &= J(\nabla \phi \times \nabla R), & \mathbf{a}^2 &= \nabla Z, \\ \mathbf{a}_3 &= J(\nabla R \times \nabla Z), & \mathbf{a}^3 &= \nabla \phi, \end{aligned}$$

where $\mathcal{J}^{-1} \equiv \nabla R \cdot \nabla Z \times \nabla \phi$ is the Jacobian of the sytem. These vectors are orthogonal but not normalized, as $|\nabla \phi| \sim R^{-1}$.

If we adopt to a local isoparametric coordinate system (s, t, ϕ) with s, t the local parameters of a region in the poloidal plane that can be mapped back unto the real space R, Z, we can write in a similar fashion,

$$\begin{aligned} \mathbf{a}_1 &= J(\nabla t \times \nabla \phi), & \mathbf{a}^1 &= \nabla s, \\ \mathbf{a}_2 &= J(\nabla \phi \times \nabla s), & \mathbf{a}^2 &= \nabla t, \\ \mathbf{a}_3 &= J(\nabla s \times \nabla t), & \mathbf{a}^3 &= \nabla \phi, \end{aligned}$$

where $\mathcal{J}^{-1} \equiv \nabla s \cdot \nabla t \times \nabla \phi$ is again the Jacobian, now of the (s, t, ϕ) sytem. In CASTOR, s is a radial coordinate that is linked to the poloidal flux by $\nabla \psi = f(s) \nabla s$. In `JOREK` the grid is structured in such a way that one can no longer distinguish between radial and poloidal coordinates. Instead they describe two directions within a discrete patch of the grid. In general, these vectors are not orthogonal, or

$$\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij}, \qquad \mathbf{a}^i \cdot \mathbf{a}^j = g^{ij},$$
$$\mathbf{a}_i \times \mathbf{a}_j = \mathcal{J}\varepsilon_{ijk}\mathbf{a}^k, \qquad \mathbf{a}^i \times \mathbf{a}^j = \frac{1}{\mathcal{J}}\varepsilon^{ijk}\mathbf{a}_k.$$

This gives rise to a metric g, with coeficients

$$\begin{aligned} g_{11} &= \frac{J^2}{R^2}|\nabla t|^2, & g^{11} &= |\nabla s|^2, \\ g_{22} &= \frac{J^2}{R^2}|\nabla s|^2, & g^{22} &= |\nabla t|^2, \\ g_{12} = g_{21} &= -\frac{J^2}{R^2}\nabla s \cdot \nabla t, & g^{12} = g^{21} &= \nabla s \cdot \nabla t, \\ g_{33} &= R^2 & g^{33} &= \frac{1}{R^2}. \end{aligned}$$

This gives rise to the Christoffel symbols as described in section B.

The gradient operator is defined as a contravariant vector (in tangent space), by

$$\nabla \equiv \mathbf{a}^1\partial_1 + \mathbf{a}^2\partial_2 + \mathbf{a}^3\partial_3.$$

The different variables can be defined in co- and contravariant form. Often this is done as follows:

$$\begin{aligned} \mathbf{A} &= A_1\mathbf{a}^1 + A_2\mathbf{a}^2 + A_3\mathbf{a}^3 = A_1\nabla s + A_2\nabla t + A_3\nabla\phi, \\ \mathbf{v} &= v^1\mathbf{a}_1 + v^2\mathbf{a}_2 + v^3\mathbf{a}_3, \\ \mathbf{B} &= B^1\mathbf{a}_1 + B^2\mathbf{a}_2 + B^3\mathbf{a}_3 \\ &= \frac{1}{J}\Big\{(\partial_2 A_3 - \partial_3 A_2)\mathbf{a}_1 + (\partial_3 A_1 - \partial_1 A_3)\mathbf{a}_2 + (\partial_1 A_2 - \partial_2 A_1)\mathbf{a}_3\Big\}. \end{aligned}$$

This implies that

$$\begin{aligned} (\mathbf{A}\times\mathbf{B})^1 &= \frac{1}{J}(A_2B_3 - A_3B_2), \\ (\mathbf{A}\times\mathbf{B})_1 &= J(A^2B^3 - A^3B^2), \\ (\nabla\times\mathbf{V})^1 &= \frac{1}{J}(\partial_2V_3 - \partial_3V_2), \end{aligned}$$

$$\begin{aligned} \nabla\Phi &= \partial_i\Phi, \\ \nabla\cdot\mathbf{V} &= \frac{1}{J}\partial_i(JV^i), \\ \mathbf{A}\cdot\nabla\mathbf{B} &= A^i\partial_i(B^j\mathbf{a}_j) = A^i(\partial_iB^j)\mathbf{a}_j + A^iB^j(\partial_i\mathbf{a}_j) = A^i(\partial_iB^j + \Gamma^j_{ik}B^k)\mathbf{a}_j, \end{aligned}$$

as $\partial_i\mathbf{a}_j \equiv \Gamma^k_{ij}\mathbf{a}_k$.

3 The Galerkin method

If a general solution of a set of coupled nonlinear partial differential equations cannot be found, a numerical solution may be obtained by resorting to the weak form of the equations. This is achieved by multiplying the equations by a testfunction f^* and integrating over a suitable domain. These testfunctions are usually chosen to be identical to the grid functions onto which the variables are projected.

After partial integration, where

$$\int \mathrm{d}^3V \mathbf{F} \cdot \nabla \mathbf{G} = \oint \mathrm{d}^2S\, \mathbf{F} \cdot \mathbf{G} - \int \mathrm{d}^3V \mathbf{G} \cdot \nabla \mathbf{F},$$

the weak formulation of the MHD equations is as follows

$$\int \mathrm{d}^3V \rho^* \partial_t \rho = \int \mathrm{d}^3V - \rho^* \nabla \cdot (\rho \mathbf{v}) - \nabla \rho^* \cdot (D \nabla \rho) + \rho^* S_\rho + \oint \mathrm{d}^2S \rho^* D(\mathbf{n} \cdot \nabla \rho), \tag{7}$$

$$\int \mathrm{d}^3V \rho \mathbf{v}^* \partial_t \mathbf{v} = \int \mathrm{d}^3V - \rho \mathbf{v}^* \cdot (\mathbf{v} \cdot \nabla) \mathbf{v} + (\rho T + \frac{1}{2} B^2) \nabla \cdot \mathbf{v}^* + \mathbf{v}^* \cdot (\mathbf{B} \cdot \nabla) \mathbf{B} - \oint \mathrm{d}^2S (\rho T + \frac{1}{2} B^2) \mathbf{n} \cdot \mathbf{v}^*, \tag{8}$$

$$\int \mathrm{d}^3V\, T^* \partial_t T = \int \mathrm{d}^3V - T^* \mathbf{v} \cdot \nabla T + (\gamma - 1) T^*\, T\, \nabla \cdot \mathbf{v} + \nabla T^* \cdot (K \nabla T) + T^* S_T + \oint \mathrm{d}^2S K T^* \mathbf{n} \cdot \nabla T, \tag{9}$$

$$\begin{aligned} \int \mathrm{d}^3V \mathbf{A}^* \partial_t \mathbf{A} &= \int \mathrm{d}^3V \mathbf{A}^* \cdot (\mathbf{v} \times (\nabla \times \mathbf{A})) - \eta \mathbf{A}^* \cdot (\nabla \times \nabla \times \mathbf{A}) \\ &= \int \mathrm{d}^3V \mathbf{A}^* \cdot (\mathbf{v} \times (\nabla \times \mathbf{A})) - \nabla \eta \cdot \mathbf{A}^* \times (\nabla \times \mathbf{A}) - \eta (\nabla \times \mathbf{A}^*) \cdot (\nabla \times \mathbf{A}) \\ &\quad + \oint \mathrm{d}^2S\, \eta\, \mathbf{n} \cdot (\mathbf{A}^* \cdot (\nabla \times \mathbf{A})). \end{aligned} \tag{10}$$

The projection of the testfunctions is chosen to be as follows:

$$\begin{aligned} \mathbf{A}^* &= \mathbf{A}^{1*} \mathbf{a}_1 + \mathbf{A}^{2*} \mathbf{a}_2 + \mathbf{A}^{3*} \mathbf{a}_3, \\ \mathbf{v}^* &= \mathbf{v}_1^* \mathbf{a}^1 + \mathbf{v}_2^* \mathbf{a}^2 + \mathbf{v}_3^* \mathbf{a}^3, \end{aligned}$$

so that e.g. $\mathbf{A}^* \cdot \mathbf{A} = A^{i*} A_i$.

3.1 Continuity equation

The continuity equation Eq. 7 is written as

$$\int \mathrm{d}^3V \rho^* \partial_t \rho = \int \mathrm{d}^3V - \rho^* \rho \nabla \cdot \mathbf{v} - \rho^* \mathbf{v} \cdot \nabla \rho$$
$$- D_\perp \nabla \rho^* \cdot \nabla \rho - (D_\parallel - D_\perp) \frac{1}{B^2} (\mathbf{B} \cdot \nabla \rho^*)(\mathbf{B} \cdot \nabla \rho)$$
$$+ \rho^* S_\rho + \oint \mathrm{d}^2 S \rho^* D (\mathbf{n} \cdot \nabla \rho),$$

where we used that

$$D \nabla \rho = D_\perp \nabla_\perp \rho + D_\parallel \nabla_\parallel \rho = D_\perp \nabla \rho + (D_\parallel - D_\perp) \nabla_\parallel \rho,$$

with $\nabla_\parallel \equiv B^{-1}(\mathbf{B} \cdot \nabla)$, and $B_0^2 = \mathbf{B} \cdot \mathbf{B} = g_{ij} B^i B^j$. The continuity equation then becomes

$$\rho^* \partial_t \rho = -\rho^* (v^1 \partial_1 \rho + v^2 \partial_2 \rho + v^3 \partial_3 \rho) - \rho^* \rho \frac{1}{J} (v^1 \partial_1 J + v^2 \partial_2 J + v^3 \partial_3 J) - \rho^* \rho (\partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3)$$
$$- D_\perp (g^{11} \partial_1 \rho^* \partial_1 \rho + g^{12} \partial_1 \rho^* \partial_2 \rho + g^{12} \partial_2 \rho^* \partial_1 \rho + g^{22} \partial_2 \rho^* \partial_2 \rho + g^{33} \partial_3 \rho^* \partial_3 \rho)$$
$$- \frac{(D_\parallel - D_\perp)}{|B|^2} (B^1 \partial_1 \rho^* + B^2 \partial_2 \rho^* + B^3 \partial_3 \rho^*)(B^1 \partial_1 \rho + B^2 \partial_2 \rho + B^3 \partial_3 \rho)$$

where the surface integral is left out. In the metric that was described before, $\partial_3 J = 0$.

3.2 Momentum equation

The momentum equation Eq. 8 is written as

$$\int \mathrm{d}^3V \rho \mathbf{v}^* \partial_t \mathbf{v} = \int \mathrm{d}^3V - \rho \mathbf{v}^* \cdot (\mathbf{v}\cdot\nabla)\mathbf{v} + (\rho T + \frac{1}{2}B^2)\nabla\cdot\mathbf{v}^* + \mathbf{v}^*\cdot(\mathbf{B}\cdot\nabla)\mathbf{B}$$
$$- \oint \mathrm{d}^2S(\rho T + \frac{1}{2}B^2)\mathbf{n}\cdot\mathbf{v}^*,$$

Then with the projections $\mathbf{v}^* \equiv v_i^* \mathbf{a}^i$ and $\mathbf{v} \equiv v^i \mathbf{a}_i$, we have that

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = (v^i\partial_i)(v^j\mathbf{a}_j) = (v^i\partial_i v^j)\mathbf{a}_j + v^i v^j(\partial_i \mathbf{a}_j) = (v^i\partial_i v^j)\mathbf{a}_j + v^i v^j \Gamma^k_{ij}\mathbf{a}_k.$$

and

$$\nabla\cdot\mathbf{v}^* = \frac{1}{J}\partial_i(Jv^{*i}) = \frac{1}{J}\partial_i(Jg^{ij}v_j^*)$$
$$= \frac{1}{J}\left\{\partial_1\Big(Jg^{11}v_1^* + Jg^{12}v_2^*\Big) + \partial_2\Big(Jg^{21}v_1^* + Jg^{22}v_2^*\Big) + \partial_3(Jg^{33}v_3^*)\right\},$$

and

$$((\mathbf{B}\cdot\nabla)\mathbf{B})^1 = B^i(\partial_i B^1) + B^i B^j(\Gamma^k_{ij}\mathbf{a}_k)^1 = B^i(\partial_i B^1) + B^i B^j \Gamma^1_{ij}$$

where e.g.

$$\partial_i B^1 = \partial_i \frac{1}{J}(\partial_2 A_3 - \partial_3 A_2) = -\frac{1}{J^2}(\partial_i J)(\partial_2 A_3 - \partial_3 A_2) + \frac{1}{J}(\partial_{i2}A_3 - \partial_{i3}A_2)$$

Also, $B^2 \equiv \mathbf{B}\cdot\mathbf{B} = \mathbf{a}_i\mathbf{a}_j B^i B^j = g_{ij}B^iB^j$. We note that $\Gamma^3_{jj} = \Gamma^3_{12} = 0$ for the present metric. Projecting the momentum equations on the test functions v_i^*,

$$\begin{aligned}
\rho v_i^* \partial_t v^i = &-\rho v_1^*(v^1\partial_1 v^1 + v^2\partial_2 v^1 + v^3\partial_3 v^1) - \rho v_1^*(v^1v^1\Gamma^1_{11} + 2v^1v^2\Gamma^1_{12} + v^2v^2\Gamma^1_{22} + v^3v^3\Gamma^1_{33})\\
&+ (\rho T + \frac{1}{2}g_{ij}B^iB^j)\frac{1}{J}\left(\partial_1(Jg^{11}v_1^*) + \partial_2(Jg^{12}v_1^*)\right)\\
&+ v_1^*(B^1\partial_1B^1 + B^2\partial_2B^1 + B^3\partial_3B^1)\\
&+ v_1^*(B^1B^1\Gamma^1_{11} + 2B^1B^2\Gamma^1_{12} + B^2B^2\Gamma^1_{22} + B^3B^3\Gamma^1_{33})\\
\\
&- \rho v_2^*(v^1\partial_1 v^2 + v^2\partial_2 v^2 + v^3\partial_3 v^2) - \rho v_2^*(v^1v^1\Gamma^2_{11} + 2v^1v^2\Gamma^2_{12} + v^2v^2\Gamma^2_{22} + v^3v^3\Gamma^2_{33})\\
&+ (\rho T + \frac{1}{2}g_{ij}B^iB^j)\frac{1}{J}\left(\partial_1(Jg^{12}v_2^*) + \partial_2(Jg^{22}v_2^*)\right)\\
&+ v_2^*(B^1\partial_1B^2 + B^2\partial_2B^2 + B^3\partial_3B^2)\\
&+ v_2^*(B^1B^1\Gamma^2_{11} + 2B^1B^2\Gamma^2_{12} + B^2B^2\Gamma^2_{22} + B^3B^3\Gamma^2_{33})\\
\\
&- \rho v_3^*(v^1\partial_1 v^3 + v^2\partial_2 v^3 + v^3\partial_3 v^3) - \rho v_3^*(2v^1v^3\Gamma^3_{13} + 2v^2v^3\Gamma^3_{23})\\
&+ (\rho T + \frac{1}{2}g_{ij}B^iB^j)\frac{1}{J}\partial_3(Jg^{33}v_3^*)\\
&+ v_3^*(B^1\partial_1B^3 + B^2\partial_2B^3 + B^3\partial_3B^3) + v_3^*(2B^1B^3\Gamma^3_{13} + 2B^2B^3\Gamma^3_{23})
\end{aligned}$$

The viscosity term is to be treated as such:

$$\begin{aligned}\int \mathrm{d}^3 V \nu \mathbf{v}^* \cdot \nabla^2 \mathbf{v} &= \int \mathrm{d}^3 V \nu \mathbf{v}^* \cdot \Big(\nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v})\Big) \\ &= -\int \mathrm{d}^3 V \nu (\nabla\cdot\mathbf{v}^*)(\nabla\cdot\mathbf{v}) - \int \mathrm{d}^3 V \nu (\nabla\times\mathbf{v}^*)\cdot(\nabla\times\mathbf{v})\end{aligned}$$

so that if we have

$$\begin{aligned}\nabla\cdot\mathbf{v}^* &= \mathbf{a}^i \partial_i g^{jk} v_k^* \mathbf{a}_j = \frac{1}{\mathcal{J}} \partial_i \mathcal{J} g^{ij} v_j^*, \\ \nabla\cdot\mathbf{v} &= \mathbf{a}^i \partial_i v^j \mathbf{a}_j = \frac{1}{\mathcal{J}} \partial_i \mathcal{J} v^i, \\ \nabla\times\mathbf{v}^* &= \frac{1}{\mathcal{J}} \varepsilon_{ijk} \partial_i v_j^*, \\ \nabla\times\mathbf{v} &= \frac{1}{\mathcal{J}} \varepsilon_{ijk} \partial_i (g_{jl} v^l)\end{aligned}$$

the viscosity terms reduce to

$$\nu \mathbf{v}^* \cdot \nabla^2 \mathbf{v} = -\nu \Big((\nabla\cdot\mathbf{v}^*)(\nabla\cdot\mathbf{v}) + (\nabla\times\mathbf{v}^*)\cdot(\nabla\times\mathbf{v}) \Big)$$

or explicitly, if

$$\begin{aligned}\nabla\times\mathbf{v}^* &= \frac{1}{\mathcal{J}} \Big(\mathbf{a}_1(\partial_2 v_3^* - \partial_3 v_2^*) + \mathbf{a}_2(\partial_3 v_1^* - \partial_1 v_3^*) + \mathbf{a}_3(\partial_1 v_2^* - \partial_2 v_1^*) \Big) \\ &= \frac{1}{\mathcal{J}} \Big((\mathbf{a}_2 \partial_3 v_1^* - \mathbf{a}_3 \partial_2 v_1^*) \\ &\quad + (\mathbf{a}_3 \partial_1 v_2^* - \mathbf{a}_1 \partial_3 v_2^*) \\ &\quad + (\mathbf{a}_1 \partial_2 v_3^* - \mathbf{a}_2 \partial_1 v_3^*) \Big), \\ \nabla\times\mathbf{v} &= \frac{1}{\mathcal{J}} \Big(\mathbf{a}_1 (\partial_2 (g_{33} v^3) - \partial_3 (g_{12} v^1 + g_{22} v^2)) \\ &\quad + \mathbf{a}_2 (\partial_3 (g_{11} v^1 + g_{12} v^2) - \partial_1 (g_{33} v^3)) \\ &\quad + \mathbf{a}_3 (\partial_1 (g_{12} v^1 + g_{22} v^2) - \partial_2 (g_{11} v^1 + g_{12} v^2)) \Big)\end{aligned}$$

then

$$\begin{aligned}(\nabla\cdot\mathbf{v}^*)(\nabla\cdot\mathbf{v}) = &\quad \frac{1}{\mathcal{J}^2} \Big(\partial_1 (\mathcal{J} g^{11} v_1^*) + \partial_2 (\mathcal{J} g^{12} v_1^*) \Big) \Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3 \Big) \\ &+ \frac{1}{\mathcal{J}^2} \Big(\partial_1 (\mathcal{J} g^{12} v_2^*) + \partial_2 (\mathcal{J} g^{22} v_2^*) \Big) \Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3 \Big) \\ &+ \frac{1}{\mathcal{J}^2} \partial_3 (\mathcal{J} g^{33} v_3^*) \Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3 \Big), \\ (\nabla\times\mathbf{v}^*)\cdot(\nabla\times\mathbf{v}) = &\frac{1}{\mathcal{J}} \Big((g_{12} (\nabla\times\mathbf{v})^1 + g_{22} (\nabla\times\mathbf{v})^2) \partial_3 v_1^* - g_{33} (\nabla\times\mathbf{v})^3 \partial_2 v_1^* \\ &\quad + g_{33} (\nabla\times\mathbf{v})^3 \partial_1 v_2^* - (g_{11} (\nabla\times\mathbf{v})^1 + g_{12} (\nabla\times\mathbf{v})^2) \partial_3 v_2^* \\ &\quad + (g_{11} (\nabla\times\mathbf{v})^1 + g_{12} (\nabla\times\mathbf{v})^2) \partial_2 v_3^* - (g_{12} (\nabla\times\mathbf{v})^1 + g_{22} (\nabla\times\mathbf{v})^2) \partial_1 v_3^* \Big)\end{aligned}$$

3.3 Temperature equation

The evolution equation of the electron temperature Eq. (9) is written as

$$
\begin{aligned}
\int \mathrm{d}^3V\, T^*\partial_t T = \int \mathrm{d}^3V & - T^*\mathbf{v}\cdot\nabla T + (\gamma-1)T^*\, T\,\nabla\cdot\mathbf{v} \\
& - K_\perp \nabla T^*\cdot\nabla T - (K_\parallel - K_\perp)\frac{1}{B^2}(\mathbf{B}\cdot\nabla T^*)(\mathbf{B}\cdot\nabla T) \\
& + T^* S_T + \oint \mathrm{d}^2 S K T^*\mathbf{n}\cdot\nabla T,
\end{aligned}
$$

which resembles the continuity equation. This leads to

$$
\begin{aligned}
T^*\partial_t T = & -T^*(v^1\partial_1 T + v^2\partial_2 T + v^3\partial_3 T) \\
& + (\gamma-1)T^* T\Big\{\frac{1}{J}(v^1\partial_1 J + v^2\partial_2 J + v^3\partial_3 J) + (\partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3)\Big\} \\
& - K_\perp(\partial_1 T^*\partial_1 T + \partial_2 T^*\partial_2 T + \partial_3 T^*\partial_3 T) \\
& - \frac{(K_\parallel - K_\perp)}{|B|^2}(B^1\partial_1 T^* + B^2\partial_2 T^* + B^3\partial_3 T^*)(B^1\partial_1 T + B^2\partial_2 T + B^3\partial_3 T)
\end{aligned}
$$

3.4 Induction equation

The induction equation Eq. 10 is written as

$$\int \mathrm{d}^3V \mathbf{A}^* \partial_t \mathbf{A} = \int \mathrm{d}^3V \mathbf{A}^* \cdot (\mathbf{v} \times (\nabla \times \mathbf{A})) - \nabla\eta \cdot \mathbf{A}^* \times (\nabla \times \mathbf{A}) - \eta(\nabla \times \mathbf{A}^*) \cdot (\nabla \times \mathbf{A})$$
$$+ \oint \mathrm{d}^2S\, \eta\, \mathbf{n} \cdot (\mathbf{A}^* \cdot (\nabla \times \mathbf{A})).$$

Then, because $\mathbf{A}^* = A^{i*}\mathbf{a}_i$, $\mathbf{A} = A_i\mathbf{a}^i$ and $\mathbf{a}_i \times \mathbf{a}_j = J\mathbf{a}^k$, we get the terms

$$\begin{aligned}
\mathbf{A}^* \cdot \partial_t \mathbf{A} &= A^{i*}\partial_t A_i = A^{*1}\partial_t A_1 + A^{*2}\partial_t A_2 + A^{*3}\partial_t A_3,\\
\mathbf{A}^* \cdot (\mathbf{v} \times (\nabla \times \mathbf{A}) &= A^{i*}\mathbf{a}_i(\mathbf{v} \times \mathbf{B})_i\mathbf{a}^i\\
&= J\Big\{A^{1*}(v^2B^3 - v^3B^2) + A^{2*}(v^3B^1 - v^1B^3) + A^{3*}(v^1B^2 - v^2B^1)\Big\}\\
\mathbf{A}^* \times (\nabla \times \mathbf{A}) &= \mathbf{A}^* \times \mathbf{B}\\
&= J\Big\{A^{1*}(B^2\mathbf{a}^3 - B^3\mathbf{a}^2) + A^{2*}(B^3\mathbf{a}^1 - B^1\mathbf{a}^3) + A^{3*}(B^1\mathbf{a}^2 - B^2\mathbf{a}^1)\Big\}\\
(\nabla\eta)_i &= \mathbf{a}^i \cdot (\mathbf{a}^j\partial_j)\eta = g^{ij}\partial_j\eta,\\
(\nabla \times \mathbf{A}^*) \cdot (\nabla \times \mathbf{A}) &= (\nabla \times \mathbf{A}^*)^i B_i = (\nabla \times \mathbf{A}^*)^i g_{ij} B^j,
\end{aligned}$$

where

$$\begin{aligned}
(\nabla \times \mathbf{A}^*)^1 &= \frac{1}{J}(\partial_2 A_3^* - \partial_3 A_2^*) = \frac{1}{J}(\partial_2(g_{33}A^{*3}) - \partial_3(g_{12}A^{1*} + g_{22}A^{2*})),\\
(\nabla \times \mathbf{A}^*)^2 &= \frac{1}{J}(\partial_3 A_1^* - \partial_1 A_3^*) = \frac{1}{J}(\partial_3(g_{12}A^{1*} + g_{22}A^{2*}) - \partial_1(g_{33}A^{3*})),\\
(\nabla \times \mathbf{A}^*)^3 &= \frac{1}{J}(\partial_1 A_2^* - \partial_2 A_1^*) = \frac{1}{J}(\partial_1(g_{12}A^{1*} + g_{22}A^{2*}) - \partial_2(g_{11}A^{1*} + g_{12}A^{2*})),\\
B_1 &= (g_{11}B^1 + g_{12}B^2),\\
B_2 &= (g_{12}B^1 + g_{22}B^2),\\
B_3 &= g_{33}B^3.
\end{aligned}$$

This leads to the projection onto A^{*i}

$$\begin{aligned}
A^{i*}\partial_t A_i =& J(v^2B^3 - v^3B^2)A^{1*} - JA^{1*}(B^2g^{33}\partial_3\eta - B^3(g^{12}\partial_1\eta + g^{22}\partial_2\eta))\\
&- \frac{\eta}{J}g_{33}B^3(\partial_1(g_{12}A^{1*}) - \partial_2(g_{11}A^{1*}))\\
&\\
&+ J(v^3B^1 - v^1B^3)A^{2*} + JA^{2*}(B^1g^{33}\partial_3\eta - B^3(g^{11}\partial_1\eta + g^{12}\partial_2\eta))\\
&- \frac{\eta}{J}g_{33}B^3(\partial_1(g_{22}A^{2*}) - \partial_2(g_{12}A^{2*}))\\
&\\
&+ J(v^1B^2 - v^2B^1)A^{3*} - JA^{3*}(B^1(g^{12}\partial_1\eta + g^{22}\partial_2\eta) - B^2(g^{11}\partial_1\eta + g^{12}\partial_2\eta))\\
&- \frac{\eta}{J}\left\{(g_{11}B^1 + g_{12}B^2)\partial_2(g_{33}A^{3*}) + (g_{12}B^1 + g_{22}B^2)\partial_1(g_{33}A^{3*})\right\}
\end{aligned}$$

4 Decomposition in JOREK

In JOREK the nonlinear initial value problem is solved using the fully implicit Crank-Nicholson by defining a state vector $\mathbf{P}(\mathbf{y})$ that depends on all the variables $\mathbf{y}$, so that

$$\partial_t \mathbf{P}(\mathbf{y}) = \mathbf{Q}(\mathbf{y}),$$

where $\mathbf{Q}$ now represents the righthandside of the MHD equations. This expression can now be linearised by writing

$$\frac{\partial \mathbf{P}}{\partial \mathbf{y}}\delta\mathbf{y} = \delta t \left(\mathbf{Q}(\mathbf{y}) + \frac{1}{2}\frac{\partial \mathbf{Q}(\mathbf{y})}{\partial \mathbf{y}}\delta\mathbf{y} \right),$$

so that

$$(\mathbf{P}' - \frac{1}{2}\mathbf{Q}'\delta t)\delta\mathbf{y} = \delta t \mathbf{Q}(\mathbf{y}).$$

The right hand side contains the information of the previous time step, whereas the left hand side contains the unknown variables.

The variables are linearized in the usual way, where $\mathbf{y} = \mathbf{y}_0 + \tilde{\mathbf{y}}$, where we now define

$$\mathbf{y} \equiv [\rho, v^1, v^2, v^3, T, A_1, A_2, A_3]^T$$

so that the first terms of the continuity equation (7) become

$$\begin{aligned}
&\left(J\rho^*\rho\delta\rho + \frac{1}{2}\delta t J\rho^*(v^1\partial_1\rho + \rho v^1\partial_1 J + \rho\partial_1 v^1) + \ldots \right) \\
= &\left(J\rho^*\tilde{\rho}\delta\rho + \frac{1}{2}\delta t J\rho^*(v_0^1\partial_1\tilde{\rho}\delta\rho + \tilde{v}^1\partial_1\rho_0\delta v^1 \right. \\
&\quad \left. + (\rho_0\tilde{v}^1\delta v^1 + \tilde{\rho}v_0^1\delta\rho)\partial_1 J + \rho_0\partial_1\tilde{v}^1\delta v^1 + \tilde{\rho}\partial_1 v_0^1\delta\rho) + \ldots \right) \\
= &\ \delta t \mathbf{Q}(\mathbf{y})
\end{aligned}$$

andsoforth for the remaining terms. The term $\mathbf{Q}(\mathbf{y})$ now contains all the terms of zeroeth order in the perturbation, e.g. $v_0^1\partial_1\rho_0$ etc. In this way we can put this in matrix notation, where

$$\delta\mathbf{y} = [\delta\rho, \delta v^1, \delta v^2, \delta v^3, \delta T, \delta A_1, \delta A_2, \delta A_3]^T = [\tilde{\rho}, \tilde{u}^1, \tilde{u}^2, \tilde{u}^3, \tilde{T}, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3]^T.$$

5 MHD equilibrium

The MHD equilibrium is constituted by the following requirements,

$$\mathbf{J} \times \mathbf{B} - \nabla p = 0, \qquad \text{Force balance,} \tag{11}$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}, \qquad \text{Induction,} \tag{12}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \text{No sources (monopoles) of magnetic field.} \tag{13}$$

and furthermore a boundary condition, e.g. that on a superconducting wall limiting the plasma the radial component of the magnetic field vanishes,

$$\mathbf{n} \cdot \mathbf{B}|_{\text{wall}} = 0.$$

5.1 Equilibrium magnetic field

The equilibrium magnetic field is by definition[1]

$$\begin{aligned}
\mathbf{B}_0 &\equiv \nabla \times \mathbf{A}_0 \\
&= (\mathbf{a}^i \partial_i) \times (A_j \mathbf{a}^j) \\
&= \partial_i A_j (\mathbf{a}^i \times \mathbf{a}^j) \\
&= \varepsilon_{ijk} \partial_i A_j \frac{1}{\mathcal{J}} \mathbf{a}_k \\
&\equiv B^k a_k
\end{aligned}$$

with the gauge $\nabla \Phi = 0$ and $\mathcal{J}$ the three dimensional Jacobian of the projection. When the equilibrium magnetic field has symmetry in the $\mathbf{a}^3$-direction (an *axisymmetric* equilibrium, so $\partial_3 \mathbf{A} = 0$) this is also written as

$$\mathbf{B}_0 = F\mathbf{a}^3 + \nabla \psi \times \mathbf{a}^3. \tag{14}$$

The components are now defined as

$$\begin{aligned}
\mathbf{B}_0^l = \mathbf{B}_0 \cdot \mathbf{a}^l &= \frac{1}{\mathcal{J}} \varepsilon_{ijk} \partial_j A_k \mathbf{a}_i \cdot \mathbf{a}^l \\
&= \frac{1}{\mathcal{J}} \varepsilon_{ljk} \partial_j A_k, \\
\text{or} \quad &= F\mathbf{a}^3 \cdot \mathbf{a}^l + \nabla \psi \times \mathbf{a}^3 \cdot \mathbf{a}^l \\
&= F g^{l3} + \mathbf{a}^3 \times \mathbf{a}^l \cdot \nabla \psi \\
&= F g^{l3} + \frac{1}{\mathcal{J}} \varepsilon_{m3l} \partial_m \psi
\end{aligned}$$

---

[1] At least if

$$\mathbf{a}^i \partial_i \times A_j \mathbf{a}^j = \partial_i A_j (\mathbf{a}^i \times \mathbf{a}^j),$$

instead of

$$\begin{aligned}
\mathbf{a}^i \partial_i \times A_j \mathbf{a}^j &= \partial_i A_j (\mathbf{a}^i \times \mathbf{a}^j) + A_j \mathbf{a}^i \times (\partial_i \mathbf{a}^j) \\
&= \partial_i A_j (\mathbf{a}^i \times \mathbf{a}^j) + A_j \mathbf{a}^i \times (\partial_i g^{jl} \mathbf{a}_l) \\
&= \partial_i A_j (\mathbf{a}^i \times \mathbf{a}^j) + A_j \mathbf{a}^i \times (\mathbf{a}_l \partial_i g^{jl} + \Gamma^k_{il} \mathbf{a}_k),
\end{aligned}$$

which would be annoying.

so that

$$\begin{aligned}
\partial_2 A_3 - \partial_3 A_2 &= \quad \partial_2 \psi, \\
\partial_3 A_1 - \partial_1 A_3 &= -\partial_1 \psi, \\
\frac{1}{\mathcal{J}} \partial_1 A_2 - \partial_2 A_1 &= g^{33} F.
\end{aligned}$$

If we assume the equilibrium vector field to have $\partial_3 A_i = 0$, we still have an underdetermined equation for A_1, A_2.

To determine the vector potential $\mathbf{A}$ from the profile F there is a freedom, as there are two components of $\mathbf{A}$ involved. One way to overcome this is to write the third component of the vector potential as a potential χ itself, or

$$\mathbf{A} = \nabla \chi \times \mathbf{a}^3, \tag{15}$$

so that

$$\begin{aligned}
A_1 &= \quad \partial_2 \chi, \\
A_2 &= -\partial_1 \chi,
\end{aligned}$$

so that the equation to determine A_1, A_2 from F becomes

$$\partial_1^2 \chi + \partial_2^2 \chi = \nabla_\perp^2 \chi = \mathcal{J} \frac{F}{R}. \tag{16}$$

This can be calculated on the local coordinates $\{s, t\}$ in `JOREK` . This does however lead to deterioration of precision of the equilibrium, as $\mathbf{A}$ now becomes a derivative of the given profile F.

Another way to resolve the issue is to simply put $A_1 = A_2 = 0$, and put

$$\mathbf{B} = F_0(\psi) g^{33} \mathbf{a}_3 + \nabla \times \mathbf{A}_0,$$

which gives $B^3 = \mathbf{B} \cdot \mathbf{a}^3 = F/R^2$, and so

$$B_0^1 = \partial_2 A_{30}, \quad B_0^2 = -\partial_1 A_{30}, \quad B_0^3 = F/R^2,$$

for the equilibrium field.

5.2 The Grad-Shafranov equation

The input necessary to compute an MHD equilibrium is a pressure profile, $\rho(\psi), T(\psi)$ and a profile of the toroidal magnetic field strength $F(\psi)$, or the better known 'current' profile FF', where a prime denotes differentiation with respect to ψ.

If we write out the force balance equation in general notation, we get

$$\begin{aligned}
\mathbf{J} \times \mathbf{B} &= (\nabla \times \mathbf{B}) \times \mathbf{B} \\
&= (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla B^2 \\
&= B^j \partial_j (B^i \mathbf{a}_i) - \mathbf{a}_i \frac{1}{2} g^{ij} \partial_j (g_{kl} B^k B^l)
\end{aligned} \tag{17}$$

where $\mathbf{J}$ can be written down explicitly as

$$\begin{aligned}
\mathbf{J} &= \nabla \times \mathbf{B} \\
&= \mathbf{a}^i \partial_i \times B_j \mathbf{a}^j \\
&= \mathbf{a}^i \times (\mathbf{a}^j \partial_i g_{jm} B^m) \\
&= \mathbf{a}^i \times \mathbf{a}^j (g_{jm} \partial_i B^m + B^m \partial_i g_{jm}) \\
&= \frac{1}{\mathcal{J}} \varepsilon^{ijk} \mathbf{a}_k (g_{jm} \partial_i B^m + B^m \partial_i g_{jm})
\end{aligned}$$

When the equilibrium magnetic field is entered in the form of

$$\mathbf{B} = F \nabla \phi + \nabla \psi \times \nabla \phi = F \mathbf{a}^3 + \mathbf{a}^i \partial_i \psi \times \mathbf{a}^3,$$

this gives the following expressions for the current:

$$\begin{aligned}
J^1 = (\nabla \times \mathbf{B}) \cdot \mathbf{a}^1 &= (\mathbf{a}^2 \partial_2 \times (\mathbf{a}^3 g_{33} B^3) \cdot \mathbf{a}^1 \\
&= \mathbf{a}^2 \partial_2 \times (\mathbf{a}^3 g_{33} g^{33} F) \cdot \mathbf{a}^1 \\
&= \quad \frac{1}{\mathcal{J}} (\partial_2 F) \\
J^2 = (\nabla \times \mathbf{B}) \cdot \mathbf{a}^2 &= (\mathbf{a}^1 \partial_1 \times (\mathbf{a}^3 g_{33} B^3) \cdot \mathbf{a}^2 \\
&= \mathbf{a}^1 \partial_1 \times (\mathbf{a}^3 g_{33} g^{33} F) \cdot \mathbf{a}^2 \\
&= -\frac{1}{\mathcal{J}} (\partial_1 F) \\
J^3 = (\nabla \times \mathbf{B}) \cdot \mathbf{a}^3 &= (\mathbf{a}^1 \partial_1 \times (\mathbf{a}^1 B_1 + \mathbf{a}^2 B_2) + \mathbf{a}^2 \partial_2 \times (\mathbf{a}^1 B_1 + \mathbf{a}^2 B_2)) \cdot \mathbf{a}^3 \\
&= \mathbf{a}^3 \cdot \Big(\mathbf{a}^1 \partial_1 \times (\mathbf{a}^2 (g_{12} B^1 + g_{22} B^2) \\
&\qquad + \mathbf{a}^2 \partial_2 \times \mathbf{a}^1 (g_{11} B^1 + g_{12} B^2)) \Big) \\
&= \frac{1}{\mathcal{J}} \partial_1 (g_{12} B^1 + g_{22} B^2) - \frac{1}{\mathcal{J}} \partial_2 (g_{11} B^1 + g_{12} B^2) \\
&= -\frac{1}{\mathcal{J}} \partial_1 (g^{12} \frac{1}{\mathcal{J}} \partial_2 A_3 + g^{11} \frac{1}{\mathcal{J}} \partial_1 A_3) - \frac{1}{\mathcal{J}} \partial_2 (g^{22} \frac{1}{\mathcal{J}} \partial_2 A_3 + g^{12} \frac{1}{\mathcal{J}} \partial_1 A_3) \\
&\equiv -\Delta^* A_3
\end{aligned}$$

where $\mathbf{a}^3 \cdot (\mathbf{a}^1 \times \mathbf{a}^2) = \mathcal{J}^{-1}$, and the unorthodox definition of the Grad-Shafranov operator $\Delta^* = \mathcal{J}^{-1} \partial_i g^{ij} \mathcal{J}^{-1} \partial_j$, or equivalently

$$\begin{aligned}
\mathbf{J} &= \nabla \times (F \nabla \phi) + \nabla \times (\nabla \psi \times \nabla \phi) \\
&= \nabla F \times \nabla \phi + \nabla \psi (\nabla \cdot \nabla \phi) + (\nabla \phi \cdot \nabla) \nabla \psi - \nabla \phi (\nabla \cdot \nabla \psi) - (\nabla \psi \cdot \nabla) \nabla \phi \\
&= \nabla F \times \nabla \phi + \nabla \phi (\nabla^2 \psi - \frac{2}{R} \partial_R \psi) \\
&= \nabla F \times \nabla \phi + \nabla \phi (R^{-1} \Delta^* \psi),
\end{aligned}$$

with $\Delta^* \equiv R \nabla R^{-1} \nabla$ the elliptic gradient operator, the same as described before as $\nabla \phi \sim R^{-1} \sim \mathcal{J}^{-1}$.

Then, the components of the $\mathbf{J}\times\mathbf{B}$ force can be specified as

$$\begin{aligned}
(\mathbf{J}\times\mathbf{B})^1 &= \mathbf{a}^1\cdot(\mathbf{J}\times\mathbf{B})\\
&= \mathbf{a}^1\cdot(\mathbf{a}_i J^i\times\mathbf{a}_j B^j)\\
&= g^{11}\mathcal{J}(J^2B^3-J^3B^2)+g^{12}\mathcal{J}(J^3B^1-J^1B^3)\\
&= g^{11}(-g^{33}F\partial_1F+J^3\partial_1A_3)+g^{12}(J^3\partial_2A_3-g^{33}F\partial_2F),\\
(\mathbf{J}\times\mathbf{B})^2 &= g^{12}\mathcal{J}(J^2B^3-J^3B^2)+g^{22}\mathcal{J}(J^3B^1-J^1B^3)\\
&= g^{12}(-g^{33}F\partial_1F+J^3\partial_1A_3)+g^{22}(J^3\partial_2A_3-g^{33}F\partial_2F),\\
(\mathbf{J}\times\mathbf{B})^3 &= g^{33}\mathcal{J}(J^1B^2-J^2B^1)\\
&= g^{33}(-\partial_2F\,\partial_1A_3+\partial_1F\,\partial_2A_3)/\mathcal{J},
\end{aligned}$$

where in the first term one recognises the Grad-Shafranov contribution that should be cancelled by the pressure gradient, and the second and third term should vanish in an MHD equilibrium. This can be seen if the coordinates $\{s,t\}$ are flux aligned, so that the flux A_3 and flux functions such as p and F have vanishing derivative with respect to t ($\partial_2\equiv 0$). It goes without saying that this formulation ought to be equivalent to the one in Eq. (17).

To make the suitable representation for ∇p, we write

$$\nabla p = \mathbf{a}^i\partial_i(\rho T) = \mathbf{a}_i g^{ij}\partial_j(\rho T)$$

so that both forces are projected onto the contravariant basis.

In `JOREK` , for given profiles $FF'(\psi), p(\psi)$, the $\psi(R,Z)$ is found by solving the Grad-Shafranov equation,

$$\Delta^*\psi = -FF'(\psi,R)-R^2p'(\psi) = RJ_\phi, \tag{18}$$

where prime denotes differentiation with respect to ψ. Note here that though F and p may be flux functions, R certainly is not.

If the current profile FF' is given, e.g. $FF'(\psi)=\sum_n f_n\psi^n$, the equilibrium toroidal magnetic field can be calculated thus

$$F(\psi)=\sqrt{2\int \mathrm{d}\,\psi FF'(\psi)}=\sqrt{2\sum_n\frac{f_n}{n+1}\psi^{n+1}+F_0},$$

where the integration constant F_0 is to be taken into account, the value of the vacuum toroidal magnetic field at the center of the plasma. The profile $F(\psi)$ will be defined in the code to give the same result.

A The metric coefficients and their derivatives

The co- and contravariant basisvectors can be defined using the local coordinate system $\{s, t, \phi\}$ of the Bezier patches, in which the gradient of the local coordinate serves as the covariant basis. In this way we arrive at a system defined as follows

$$\begin{aligned} \mathbf{a}_1 &= \mathcal{J}(\nabla t \times \nabla \phi), & \mathbf{a}^1 &= \nabla s, \\ \mathbf{a}_2 &= \mathcal{J}(\nabla \phi \times \nabla s), & \mathbf{a}^2 &= \nabla t, \\ \mathbf{a}_3 &= \mathcal{J}(\nabla s \times \nabla t), & \mathbf{a}^3 &= \nabla \phi, \end{aligned}$$

where $\mathcal{J}^{-1} \equiv \nabla s \times \nabla t \cdot \nabla \phi$ the three dimensional Jacobian of the system, [2] so that

$$\begin{aligned} g_{11} &= \mathcal{J}^2 |\nabla \phi|^2 |\nabla t|^2, & g^{11} &= |\nabla s|^2, \\ g_{22} &= \mathcal{J}^2 |\nabla \phi|^2 |\nabla s|^2, & g^{22} &= |\nabla t|^2, \\ g_{12} = g_{21} &= -\mathcal{J}^2 |\nabla \phi|^2 \nabla s \cdot \nabla t, & g^{12} = g^{21} &= \nabla s \cdot \nabla t, \\ g_{33} &= \mathcal{J}^2 |\nabla s|^2 |\nabla t|^2 = |\nabla \phi|^{-2} = R^2 & g^{33} &= \frac{1}{R^2}. \end{aligned}$$

The magnitude of the vectors $h_i \equiv |\mathbf{a}_i|$ can be calculated by using the fact that $\mathbf{a}_i = g_{ij}\mathbf{a}^j$,

$$\begin{aligned} |\mathbf{a}_1| &= \sqrt{g_{11}} = |g_{11}\mathbf{a}^1 + g_{12}\mathbf{a}^2| = \sqrt{3g_{11} + g_{12}^2 g^{22}}, & |\mathbf{a}^1| &= |\nabla s| = \sqrt{g^{11}}, \\ |\mathbf{a}_2| &= \sqrt{g_{22}} = |g_{11}\mathbf{a}^1 + g_{12}\mathbf{a}^2| = \sqrt{3g_{22} + g_{12}^2 g^{11}}, & |\mathbf{a}^2| &= |\nabla t| = \sqrt{g^{22}}, \\ |\mathbf{a}_3| &= \sqrt{g_{33}} = |g_{33}\mathbf{a}^3|, & |\mathbf{a}^3| &= |\nabla \phi| = \sqrt{g^{33}}, \end{aligned}$$

The following identities apply:

$$\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} s_x & t_x \\ s_y & t_y \end{pmatrix} \begin{pmatrix} \psi_s \\ \psi_t \end{pmatrix}$$

$$\begin{pmatrix} \psi_s \\ \psi_t \end{pmatrix} = \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

so that

$$\begin{pmatrix} s_x & t_x \\ s_y & t_y \end{pmatrix} = \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}^{-1}$$

and vice versa, or

$$\begin{pmatrix} s_x & t_x \\ s_y & t_y \end{pmatrix} = \frac{1}{x_s y_t - x_t y_s} \begin{pmatrix} y_t & -y_s \\ -x_t & x_s \end{pmatrix}$$

$$\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} = \frac{1}{s_x t_y - s_y t_x} \begin{pmatrix} t_y & -t_x \\ -s_y & s_x \end{pmatrix}$$

[2] or, in our case, $\mathcal{J} \equiv JR$, with J the Jacobian of the two-dimensional subsystem spanned by $\{s, t\}$,

where we identify $J \equiv x_s y_t - x_t y_s$ as the Jacobian of the 2-D system $\{s, t\}$. This allows us to rewrite the following derivatives of the metric coefficients

$$
\begin{aligned}
\partial_1 g_{11} &= \partial_s J^2 |\nabla t|^2 = \partial_s J^2 (t_x^2 + t_y^2) = \partial_s J^2 \left(\frac{1}{J^2} ((-y_s)^2 + x_s^2) \right) \\
&= 2x_s x_{ss} + 2y_s y_{ss}, \\
\partial_1 g_{22} &= \partial_s J^2 |\nabla s|^2 = \partial_s J^2 (s_x^2 + s_y^2) = \partial_s J^2 \left(\frac{1}{J^2} (y_t^2 + (-x_t)^2) \right) \\
&= 2x_t x_{st} + 2y_t y_{st}, \\
\partial_2 g_{11} &= \partial_t J^2 |\nabla t|^2 = \partial_t J^2 (t_x^2 + t_y^2) = \partial_t J^2 \left(\frac{1}{J^2} ((-y_s)^2 + x_s^2) \right) \\
&= 2x_s x_{st} + 2y_s y_{st}, \\
\partial_2 g_{22} &= \partial_t J^2 |\nabla s|^2 = \partial_t J^2 (s_x^2 + s_y^2) = \partial_t J^2 \left(\frac{1}{J^2} (y_t^2 + (-x_t)^2) \right) \\
&= 2x_t x_{tt} + 2y_t y_{tt}, \\
\partial_1 g_{12} &= \partial_s - J^2 \nabla s \cdot \nabla t = -\partial_s J^2 (s_x t_x + s_y t_y) = \partial_s J^2 \left(\frac{1}{J^2} (y_s y_t + x_s x_t) \right) \\
&= x_s x_{st} + x_{ss} x_t + y_s y_{st} + y_{ss} y_t, \\
\partial_2 g_{12} &= \partial_t - J^2 \nabla s \cdot \nabla t = -\partial_t J^2 (s_x t_x + s_y t_y) = \partial_t J^2 \left(\frac{1}{J^2} (y_s y_t + x_s x_t) \right) \\
&= x_s x_{tt} + x_{st} x_t + y_s y_{tt} + y_{st} y_t, \\
\partial_1 g_{33} &= \partial_s R^2 = 2R \partial_s R, \\
\partial_2 g_{33} &= \partial_t R^2 = 2R \partial_t R,
\end{aligned}
$$

where we note that $x \equiv R$ in `JOREK` .

B Christoffel symbols

The Christoffel symbol of the second kind is a measure of how the metric changes in space, and is defined as $\partial_i \mathbf{a}_j = \Gamma^k_{ij}\mathbf{a}_k$, or

$$\Gamma^k_{ij} \equiv \frac{1}{2} g^{mk}(\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}).$$

For a metric g with $g_{13} = g_{31} = g_{23} = g_{32} = 0$ and $g_{12} = g_{21}$, we have

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2J^2}\left(g_{22}\partial_1 g_{11} + g_{12}\partial_2 g_{11} - 2g_{12}\partial_1 g_{12}\right)\\
\Gamma^1_{12} &= \frac{1}{2J^2}\left(g_{22}\partial_2 g_{11} - g_{12}\partial_1 g_{22}\right)\\
\Gamma^1_{22} &= \frac{1}{2J^2}\left(2g_{22}\partial_2 g_{12} - g_{22}\partial_1 g_{22} - g_{12}\partial_2 g_{22}\right)\\
\\
\Gamma^2_{11} &= \frac{1}{2J^2}\left(-g_{12}\partial_1 g_{11} - g_{11}\partial_2 g_{11} + 2g_{11}\partial_1 g_{12}\right)\\
\Gamma^2_{12} &= \frac{1}{2J^2}\left(-g_{12}\partial_2 g_{11} + g_{11}\partial_1 g_{22}\right)\\
\Gamma^2_{22} &= \frac{1}{2J^2}\left(g_{12}\partial_1 g_{22} - 2g_{12}\partial_2 g_{12} + g_{11}\partial_2 g_{22}\right)\\
\\
\Gamma^1_{33} &= \frac{1}{2J^2}\left(-g_{22}\partial_1 g_{33} + g_{12}\partial_2 g_{33}\right) = \frac{R}{J^2}\left(-g_{22}\partial_s R + g_{12}\partial_t R\right)\\
\Gamma^2_{33} &= \frac{1}{2J^2}\left(g_{12}\partial_1 g_{33} - g_{11}\partial_2 g_{33}\right) = \frac{R}{J^2}\left(g_{12}\partial_s R - g_{11}\partial_t R\right)\\
\Gamma^3_{13} &= \frac{1}{2g_{33}}\partial_1 g_{33} = \frac{1}{R}\partial_s R,\\
\Gamma^3_{23} &= \frac{1}{2g_{33}}\partial_2 g_{33} = \frac{1}{R}\partial_t R,
\end{aligned}$$

where it was used that

$$\begin{aligned}
g_{11} &= \mathcal{J}^2 g^{33} g^{22} = \frac{J^2R^2}{R^2} g^{22} = J^2 g^{22},\\
g_{22} &= \mathcal{J}^2 g^{33} g^{11} = \frac{J^2R^2}{R^2} g^{11} = J^2 g^{11},\\
g_{12} &= -\mathcal{J}^2 g^{33} g^{12} = -\frac{J^2R^2}{R^2} g^{12} = -J^2 g^{12},\\
g_{33} &= R^2 = (g^{33})^{-1}.
\end{aligned}$$

These expressions may be checked using the identity

$$\frac{\partial g_{jk}}{\partial x_i} = g_{kl}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ki}.$$

This gives

$$
\begin{aligned}
\partial_1 g_{11} &= 2g_{11}\Gamma^1_{11} + 2g_{12}\Gamma^2_{11},\\
\partial_2 g_{11} &= 2g_{11}\Gamma^1_{12} + 2g_{12}\Gamma^2_{12},\\
\partial_1 g_{12} &= g_{12}\Gamma^1_{11} + g_{22}\Gamma^2_{11} + g_{11}\Gamma^1_{12} + g_{12}\Gamma^2_{12},\\
\partial_2 g_{12} &= g_{12}\Gamma^1_{12} + g_{22}\Gamma^2_{12} + g_{11}\Gamma^1_{22} + g_{12}\Gamma^2_{22},\\
\partial_1 g_{22} &= 2g_{12}\Gamma^1_{12} + 2g_{22}\Gamma^2_{12},\\
\partial_2 g_{22} &= 2g_{12}\Gamma^1_{22} + 2g_{22}\Gamma^2_{22}.
\end{aligned}
$$

C Matrix elements for `JOREK` routine to compute contribution based on MHD elements

C.1 Definition of the matrix elements

In the Crank-Nicholson paradigm with

$$\left(\mathbf{P}' - \frac{1}{2}\delta t\mathbf{Q}'\right)\delta\mathbf{y} = \mathbf{Q}\delta t,$$

where both sides of the equation need to be multiplied by $\mathrm{d}^3V = J\mathrm{d}s\mathrm{d}tR\mathrm{d}\phi$ to obtain the Galerkin (weak) formulation of the initial value problem, we thus write the components $\int P_i'\mathcal{J}\mathrm{d}s\mathrm{d}t\mathrm{d}\phi$, etc.

C.2 The $\mathbf{J} \times \mathbf{B}$ term

If we define the derivatives of the equilibrium magnetic field as

$$\partial_1 B_0^1 = \frac{1}{\mathcal{J}}(\partial_{12}A_{30} - \partial_{13}A_{20}) - \frac{1}{\mathcal{J}}(\partial_2 A_{30} - \partial_3 A_{20})\frac{\mathcal{J}_s}{\mathcal{J}}$$

and the components of the current $\mathbf{J} \equiv \nabla \times \mathbf{B}$ as

$$\begin{aligned}
J_0^1 = \frac{1}{\mathcal{J}}\Big(&\partial_2 F + g_{33}\frac{1}{\mathcal{J}}(\partial_{12}A_{20} - \partial_{22}A_{10}) \\
&- \frac{\mathcal{J}_t}{\mathcal{J}}g_{33}\frac{1}{\mathcal{J}}(\partial_1 A_{20} - \partial_2 A_{10}) + (\partial_2 g_{33})\frac{1}{\mathcal{J}}(\partial_1 A_{20} - \partial_2 A_{10}) \\
&- (g_{12}\partial_3 B_0^1 + B_0^1 \partial_3 g_{12}) \\
&- (g_{22}\partial_3 B_0^2 + B_0^2 \partial_3 g_{22})\Big) \\
J_0^2 = \frac{1}{\mathcal{J}}\Big(&-\partial_1 F - g_{33}\frac{1}{\mathcal{J}}(\partial_{11}A_{20} - \partial_{12}A_{10}) \\
&+ \frac{\mathcal{J}_s}{\mathcal{J}}g_{33}\frac{1}{\mathcal{J}}(\partial_1 A_{20} - \partial_2 A_{10}) - (\partial_s g_{33})\frac{1}{\mathcal{J}}(\partial_1 A_{20} - \partial_2 A_{10}) \\
&+ (g_{11}\partial_3 B_0^1 + B_0^1 \partial_3 g_{11}) \\
&+ (g_{12}\partial_3 B_0^2 + B_0^2 \partial_3 g_{12})\Big) \\
J_0^3 = \frac{1}{\mathcal{J}}\Big(&(g_{12}\partial_1 B_0^1 + B_0^1 \partial_1 g_{12}) \\
&+ (g_{22}\partial_1 B_0^2 + B_0^2 \partial_1 g_{22}) \\
&- (g_{11}\partial_2 B_0^1 + B_0^1 \partial_2 g_{11}) \\
&- (g_{12}\partial_2 B_0^2 + B_0^2 \partial_2 g_{12})\Big)
\end{aligned}$$

Furthermore, in order to obtain $\mathbf{J} \times \mathbf{B} \equiv (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla|\mathbf{B}^2|$ we define

$$\begin{aligned}
(\mathbf{B} \cdot \nabla)\mathbf{B}^1 &= B^1\partial_1 B^1 + B^2\partial_2 B^1 + B^3\partial_3 B^1 + B^1B^1\Gamma^1_{11} + 2B^1B^2\Gamma^1_{12} + B^2B^2\Gamma^1_{22} + B^3B^3\Gamma^1_{33}, \\
(\mathbf{B} \cdot \nabla)\mathbf{B}^2 &= B^1\partial_1 B^2 + B^2\partial_2 B^2 + B^3\partial_3 B^2 + B^1B^1\Gamma^2_{11} + 2B^1B^2\Gamma^2_{12} + B^2B^2\Gamma^2_{22} + B^3B^3\Gamma^2_{33}, \\
(\mathbf{B} \cdot \nabla)\mathbf{B}^3 &= B^1\partial_1 B^3 + B^2\partial_2 B^3 + B^3\partial_3 B^3 + 2B^1B^3\Gamma^3_{13} + 2B^2B^3\Gamma^3_{23}
\end{aligned}$$

and

$$\begin{aligned}
\partial_1|\mathbf{B}|^2 &= \partial_1(g_{11}B^1B^1 + 2g_{12}B^1B^2 + g_{22}B^2B^2 + g_{33}B^3B^3) \\
\partial_2|\mathbf{B}|^2 &= \partial_2(g_{11}B^1B^1 + 2g_{12}B^1B^2 + g_{22}B^2B^2 + g_{33}B^3B^3) \\
\partial_3|\mathbf{B}|^2 &= \partial_3(g_{11}B^1B^1 + 2g_{12}B^1B^2 + g_{22}B^2B^2 + g_{33}B^3B^3)
\end{aligned}$$

so that

$$\begin{aligned}
(\mathbf{J} \times \mathbf{B})^1 &= (\mathbf{B} \cdot \nabla)\mathbf{B}^1 - \frac{1}{2}(g^{11}\partial_1|\mathbf{B}|^2 + g^{12}\partial_2|\mathbf{B}|^2) \\
(\mathbf{J} \times \mathbf{B})^2 &= (\mathbf{B} \cdot \nabla)\mathbf{B}^2 - \frac{1}{2}(g^{12}\partial_1|\mathbf{B}|^2 + g^{22}\partial_2|\mathbf{B}|^2) \\
(\mathbf{J} \times \mathbf{B})^3 &= (\mathbf{B} \cdot \nabla)\mathbf{B}^3 - \frac{1}{2}g^{33}\partial_3|\mathbf{B}|^2
\end{aligned}$$

When we project this onto the different components of the vector potential:

$$
\begin{aligned}
(\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_1) &= \frac{1}{\mathcal{J}}(\partial_3 A_1)\partial_2 B_0^1 + \frac{1}{\mathcal{J}}(-\partial_2 A_1)\partial_3 B_0^1 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_3 A_1)\Big(B_0^1\Gamma_{12}^1 + B_0^2\Gamma_{22}^1\Big) + \frac{1}{\mathcal{J}}(-\partial_2 A_1)B_0^3\Gamma_{33}^1\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_2) &= \partial_1(-\frac{1}{\mathcal{J}}\partial_3 A_2) + \partial_2(-\frac{1}{\mathcal{J}}\partial_3 A_2) + \partial_3(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
&\quad + \frac{1}{\mathcal{J}}(-\partial_3 A_2)\partial_1 B_0^1 + \frac{1}{\mathcal{J}}(\partial_1 A_2)\partial_3 B_0^1 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(-\partial_3 A_2)\Big(B_0^1\Gamma_{11}^1 + B_0^2\Gamma_{12}^1\Big) + \frac{1}{\mathcal{J}}(\partial_1 A_2)B_0^3\Gamma_{33}^1\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_3) &= \partial_1(\frac{1}{\mathcal{J}}\partial_2 A_3) + \partial_2(\frac{1}{\mathcal{J}}\partial_2 A_3) + \partial_3(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_2 A_3)\partial_1 B_0^1 + \frac{1}{\mathcal{J}}(-\partial_1 A_3)\partial_2 B_0^1 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_2 A_3)\Big(B_0^1\Gamma_{11}^1 + B_0^2\Gamma_{12}^1\Big) + \frac{1}{\mathcal{J}}(-\partial_1 A_3)\Big(B_0^1\Gamma_{12}^1 + B_0^2\Gamma_{22}^1\Big)\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^2(A_1) &= \partial_1(\frac{1}{\mathcal{J}}\partial_3 A_1) + \partial_2(\frac{1}{\mathcal{J}}\partial_3 A_1) + \partial_3(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_3 A_1)\partial_2 B_0^2 + \frac{1}{\mathcal{J}}(-\partial_2 A_1)\partial_3 B_0^2 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_3 A_1)\Big(B_0^1\Gamma_{12}^2 + B_0^2\Gamma_{22}^2\Big) + \frac{1}{\mathcal{J}}(-\partial_2 A_1)B_0^3\Gamma_{33}^2\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^2(A_2) &= \frac{1}{\mathcal{J}}(-\partial_3 A_2)\partial_1 B_0^2 + \frac{1}{\mathcal{J}}(\partial_1 A_2)\partial_3 B_0^2 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(-\partial_3 A_2)\Big(B_0^1\Gamma_{11}^2 + B_0^2\Gamma_{12}^2\Big) + \frac{1}{\mathcal{J}}(\partial_1 A_2)B_0^3\Gamma_{33}^2\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^2(A_3) &= \partial_1(-\frac{1}{\mathcal{J}}\partial_1 A_3) + \partial_2(-\frac{1}{\mathcal{J}}\partial_1 A_3) + \partial_3(-\frac{1}{\mathcal{J}}\partial_1 A_3) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_2 A_3)\partial_1 B_0^2 + \frac{1}{\mathcal{J}}(-\partial_1 A_3)\partial_2 B_0^2 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_2 A_3)\Big(B_0^1\Gamma_{11}^2 + B_0^2\Gamma_{12}^2\Big) + \frac{1}{\mathcal{J}}(-\partial_1 A_3)\Big(B_0^1\Gamma_{12}^2 + B_0^2\Gamma_{22}^2\Big)\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^3(A_1) &= \partial_1(-\frac{1}{\mathcal{J}}\partial_2 A_1) + \partial_2(-\frac{1}{\mathcal{J}}\partial_2 A_1) + \partial_3(-\frac{1}{\mathcal{J}}\partial_2 A_1) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_3 A_1)\partial_2 B_0^3 + \frac{1}{\mathcal{J}}(-\partial_2 A_1)\partial_3 B_0^3 \\
&\quad + 2\Big(-\frac{1}{\mathcal{J}}(\partial_2 A_1)\Big(B_0^1\Gamma_{13}^3 + B_0^2\Gamma_{23}^3\Big) + \frac{1}{\mathcal{J}}(\partial_3 A_1)B_0^3\Gamma_{23}^3\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^3(A_2) &= \partial_1(\frac{1}{\mathcal{J}}\partial_1 A_2) + \partial_2(\frac{1}{\mathcal{J}}\partial_1 A_2) + \partial_3(\frac{1}{\mathcal{J}}\partial_1 A_2) \\
&\quad \frac{1}{\mathcal{J}}(-\partial_3 A_2)\partial_1 B_0^3 + \frac{1}{\mathcal{J}}(\partial_1 A_2)\partial_3 B_0^3 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_1 A_2)\Big(B_0^1\Gamma_{13}^3 + B_0^2\Gamma_{23}^3\Big) + \frac{1}{\mathcal{J}}(-\partial_3 A_2)B_0^3\Gamma_{13}^3\Big) \\
(\mathbf{B}\cdot\nabla)\mathbf{B}^3(A_3) &= \frac{1}{\mathcal{J}}(\partial_2 A_3)\partial_1 B_0^3 + \frac{1}{\mathcal{J}}(-\partial_1 A_3)\partial_2 B_0^3 \\
&\quad + 2\Big(\frac{1}{\mathcal{J}}(\partial_2 A_3)B_0^3\Gamma_{13}^3 + \frac{1}{\mathcal{J}}(-\partial_1 A_3)B_0^2\Gamma_{23}^3\Big)
\end{aligned}
$$

and likewise

$$
\begin{aligned}
\partial_1|\mathbf{B}^2|(A_1) = \quad & 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_1 g_{12} B_0^1 + 2g_{12}B_0^1\partial_1(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_1 g_{22} B_0^2 + 2g_{22}B_0^2\partial_1(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_2 A_1)\partial_1 g_{33} B_0^3 + 2g_{33}B_0^3\partial_1(-\frac{1}{\mathcal{J}}\partial_2 A_1) \\
\partial_1|\mathbf{B}^2|(A_2) = \quad & 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_1 g_{11} B_0^1 + 2g_{11}B_0^1\partial_1(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_1 g_{12} B_0^2 + 2g_{12}B_0^2\partial_1(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(\frac{1}{\mathcal{J}}\partial_1 A_2)\partial_1 g_{33} B_0^3 + 2g_{33}B_0^3\partial_1(\frac{1}{\mathcal{J}}\partial_1 A_2) \\
\partial_1|\mathbf{B}^2|(A_3) = \quad & 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_1 g_{11} B_0^1 + 2g_{11}B_0^1\partial_1(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_1 g_{12} B_0^2 + 2g_{12}B_0^2\partial_1(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_1 g_{12} B_0^1 + 2g_{12}B_0^1\partial_1(-\frac{1}{\mathcal{J}}\partial_1 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_1 g_{22} B_0^2 + 2g_{22}B_0^2\partial_1(-\frac{1}{\mathcal{J}}\partial_1 A_3)
\end{aligned}
$$

$$
\begin{aligned}
\partial_2|\mathbf{B}^2|(A_1) = \quad & 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_2 g_{12} B_0^1 + 2g_{12}B_0^1\partial_2(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_2 g_{22} B_0^2 + 2g_{22}B_0^2\partial_2(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_2 A_1)\partial_2 g_{33} B_0^3 + 2g_{33}B_0^3\partial_2(-\frac{1}{\mathcal{J}}\partial_2 A_1) \\
\partial_2|\mathbf{B}^2|(A_2) = \quad & 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_2 g_{11} B_0^1 + 2g_{11}B_0^1\partial_2(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_2 g_{12} B_0^2 + 2g_{12}B_0^2\partial_2(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(\frac{1}{\mathcal{J}}\partial_1 A_2)\partial_2 g_{33} B_0^3 + 2g_{33}B_0^3\partial_2(\frac{1}{\mathcal{J}}\partial_1 A_2) \\
\partial_2|\mathbf{B}^2|(A_3) = \quad & 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_2 g_{11} B_0^1 + 2g_{11}B_0^1\partial_2(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_2 g_{12} B_0^2 + 2g_{12}B_0^2\partial_2(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_2 g_{12} B_0^1 + 2g_{12}B_0^1\partial_2(-\frac{1}{\mathcal{J}}\partial_1 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_2 g_{22} B_0^2 + 2g_{22}B_0^2\partial_2(-\frac{1}{\mathcal{J}}\partial_1 A_3)
\end{aligned}
$$

$$
\begin{aligned}
\partial_3|\mathbf{B}^2|(A_1) = \quad & 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_3 g_{12}B_0^1 + 2g_{12}B_0^1\partial_3(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(\frac{1}{\mathcal{J}}\partial_3 A_1)\partial_3 g_{22}B_0^2 + 2g_{22}B_0^2\partial_3(\frac{1}{\mathcal{J}}\partial_3 A_1) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_2 A_1)\partial_3 g_{33}B_0^3 + 2g_{33}B_0^3\partial_3(-\frac{1}{\mathcal{J}}\partial_2 A_1) \\
\partial_3|\mathbf{B}^2|(A_2) = \quad & 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_3 g_{11}B_0^1 + 2g_{11}B_0^1\partial_3(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_3 A_2)\partial_3 g_{12}B_0^2 + 2g_{12}B_0^2\partial_3(-\frac{1}{\mathcal{J}}\partial_3 A_2) \\
& + 2(\frac{1}{\mathcal{J}}\partial_1 A_2)\partial_3 g_{33}B_0^3 + 2g_{33}B_0^3\partial_3(\frac{1}{\mathcal{J}}\partial_1 A_2) \\
\partial_3|\mathbf{B}^2|(A_3) = \quad & 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_3 g_{11}B_0^1 + 2g_{11}B_0^1\partial_3(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(\frac{1}{\mathcal{J}}\partial_2 A_3)\partial_3 g_{12}B_0^2 + 2g_{12}B_0^2\partial_3(\frac{1}{\mathcal{J}}\partial_2 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_3 g_{12}B_0^1 + 2g_{12}B_0^1\partial_3(-\frac{1}{\mathcal{J}}\partial_1 A_3) \\
& + 2(-\frac{1}{\mathcal{J}}\partial_1 A_3)\partial_3 g_{22}B_0^2 + 2g_{22}B_0^2\partial_3(-\frac{1}{\mathcal{J}}\partial_1 A_3)
\end{aligned}
$$

so that e.g.

$$
\begin{aligned}
Q'_{2,6} &= (\mathbf{J}\times\mathbf{B})^1(A_1) \\
&\equiv (\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_1) - \frac{1}{2}(g^{11}\partial_1|\mathbf{B}^2|(A_1) + g^{12}\partial_2|\mathbf{B}^2|(A_1)) \\
Q'_{2,7} &= (\mathbf{J}\times\mathbf{B})^1(A_2) \\
&\equiv (\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_2) - \frac{1}{2}(g^{11}\partial_1|\mathbf{B}^2|(A_2) + g^{12}\partial_2|\mathbf{B}^2|(A_2)) \\
Q'_{2,6} &= (\mathbf{J}\times\mathbf{B})^1(A_3) \\
&\equiv (\mathbf{B}\cdot\nabla)\mathbf{B}^1(A_3) - \frac{1}{2}(g^{11}\partial_1|\mathbf{B}^2|(A_3) + g^{12}\partial_2|\mathbf{B}^2|(A_3)).
\end{aligned}
$$

For the case that we do the partial integration to remove an order of differentiation of A_i,

$$
\begin{aligned}
|B_0|^2(A_1) &= 2g_{12}B_0^1\frac{1}{\mathcal{J}}\partial_3 A_1 + 2g_{22}B_0^2\frac{1}{\mathcal{J}}\partial_3 A_1 + 2g_{33}B_0^3(-\frac{1}{\mathcal{J}}\partial_2 A_1) \\
|B_0|^2(A_2) &= 2g_{11}B_0^1(-\frac{1}{\mathcal{J}}\partial_3 A_2) + 2g_{12}B_0^2(-\frac{1}{\mathcal{J}}\partial_3 A_2) + 2g_{33}B_0^3(\frac{1}{\mathcal{J}}\partial_1 A_2) \\
|B_0|^2(A_3) &= 2g_{11}B_0^1(\frac{1}{\mathcal{J}}\partial_2 A_3) + 2g_{12}\frac{1}{\mathcal{J}}(B_0^2\partial_2 A_3 - B_0^1\partial_1 A_3) + 2g_{22}B_0^2(-\frac{1}{\mathcal{J}}\partial_1 A_3)
\end{aligned}
$$

Note that this is also $(\partial_{A_i}|B_0|^2)\tilde{A}_i$.

C.3 The viscosity terms

As stated in section 3.2, the viscosity term can be integrated by parts to yield

$$\nu \int_\Omega \mathrm{d}^3 V \mathbf{v}^* \cdot \nabla^2 \mathbf{v} = -\nu \int_\Omega \mathrm{d}^3 V (\nabla \cdot \mathbf{v}^*)(\nabla \cdot \mathbf{v}) + (\nabla \times \mathbf{v}^*) \cdot (\nabla \times \mathbf{v})$$

so that when we split the products into terms containing the different v_i^*,

$$\begin{aligned}
\nabla \cdot \mathbf{v}^* &= (\nabla \cdot \mathbf{v}^*)(i) \\
&= \quad \frac{1}{\mathcal{J}}(\partial_1 \mathcal{J} g^{11} v_1^* + \partial_2 \mathcal{J} g^{12} v_1^*) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_1 \mathcal{J} g^{12} v_2^* + \partial_2 \mathcal{J} g^{22} v_2^*) \\
&\quad + \frac{1}{\mathcal{J}}(\partial_3 \mathcal{J} g^{33} v_3^*),
\end{aligned}$$

and

$$\begin{aligned}
\nabla \times \mathbf{v}^* &= \frac{1}{\mathcal{J}}\Big(\mathbf{a}_1(\partial_2 v_3^* - \partial_3 v_2^*) + \mathbf{a}_2(\partial_3 v_1^* - \partial_1 v_3^*) + \mathbf{a}_3(\partial_1 v_2^* - \partial_2 v_1^*)\Big) \\
&= \frac{1}{\mathcal{J}}\Big((\mathbf{a}_2 \partial_3 v_1^* - \mathbf{a}_3 \partial_2 v_1^*) \\
&\quad + (\mathbf{a}_3 \partial_1 v_2^* - \mathbf{a}_1 \partial_3 v_2^*) \\
&\quad + (\mathbf{a}_1 \partial_2 v_3^* - \mathbf{a}_2 \partial_1 v_3^*)\Big), \\
\nabla \times \mathbf{v} &= \frac{1}{\mathcal{J}}\Big(\mathbf{a}_1(\partial_2(g_{33} v^3) - \partial_3(g_{12} v^1 + g_{22} v^2)) \\
&\quad + \mathbf{a}_2(\partial_3(g_{11} v^1 + g_{12} v^2) - \partial_1(g_{33} v^3)) \\
&\quad + \mathbf{a}_3(\partial_1(g_{12} v^1 + g_{22} v^2) - \partial_2(g_{11} v^1 + g_{12} v^2))\Big)
\end{aligned}$$

then the components of right hand side (zeroeth order) of the equation become

$$\begin{aligned}
(\nabla \cdot \mathbf{v}^*)(\nabla \cdot \mathbf{v}) = &\quad \frac{1}{\mathcal{J}^2}\Big(\partial_1(\mathcal{J} g^{11} v_1^*) + \partial_2(\mathcal{J} g^{12} v_1^*)\Big)\Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3\Big) \\
&+ \frac{1}{\mathcal{J}^2}\Big(\partial_1(\mathcal{J} g^{12} v_2^*) + \partial_2(\mathcal{J} g^{22} v_2^*)\Big)\Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3\Big) \\
&+ \frac{1}{\mathcal{J}^2}\partial_3(\mathcal{J} g^{33} v_3^*)\Big(\partial_1 \mathcal{J} v^1 + \partial_2 \mathcal{J} v^2 + \partial_3 \mathcal{J} v^3\Big), \\
(\nabla \times \mathbf{v}^*) \cdot (\nabla \times \mathbf{v}) = &\ \frac{1}{\mathcal{J}}\Big((g_{12}(\nabla \times \mathbf{v})^1 + g_{22}(\nabla \times \mathbf{v})^2)\partial_3 v_1^* - g_{33}(\nabla \times \mathbf{v})^3 \partial_2 v_1^* \\
&+ g_{33}(\nabla \times \mathbf{v})^3 \partial_1 v_2^* - (g_{11}(\nabla \times \mathbf{v})^1 + g_{12}(\nabla \times \mathbf{v})^2)\partial_3 v_2^* \\
&+ (g_{11}(\nabla \times \mathbf{v})^1 + g_{12}(\nabla \times \mathbf{v})^2)\partial_2 v_3^* - (g_{12}(\nabla \times \mathbf{v})^1 + g_{22}(\nabla \times \mathbf{v})^2)\partial_1 v_3^*\Big)
\end{aligned}$$

The terms in the linearised matrix Q'_{ij} become

$$\begin{aligned}
\nu_{2,2} = \quad & (\nabla \cdot \mathbf{v}^*)(2)(\partial_1 u^1 + u^1 \mathcal{J}_s/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{12}\partial_3(g_{12}u^1) + g_{22}\partial_3(g_{11}u^1))\partial_3 u_1^* \\
& \quad - \mathcal{J}^{-2} g_{33}(\partial_1(g_{12}u^1) - \partial_2(g_{11}u^1))\partial_2 u_1^* \Big) \\
\nu_{2,3} = \quad & (\nabla \cdot \mathbf{v}^*)(2)(\partial_2 u^2 + u^2 \mathcal{J}_t/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{12}\partial_3(g_{22}u^2) + g_{22}\partial_3(g_{12}u^2))\partial_3 u_1^* \\
& \quad - \mathcal{J}^{-2} g_{33}(\partial_1(g_{22}u^2) - \partial_2(g_{12}u^2))\partial_2 u_1^* \Big) \\
\nu_{2,4} = \quad & (\nabla \cdot \mathbf{v}^*)(2)(\partial_3 u^3 + u^3 \mathcal{J}_\phi/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{12}\partial_2(g_{33}u^3) + g_{22}\partial_1(g_{33}u^3))\partial_3 u_1^* \Big) \\
\nu_{3,2} = \quad & (\nabla \cdot \mathbf{v}^*)(3)(\partial_1 u^1 + u^1 \mathcal{J}_s/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2} g_{33}(\partial_1(g_{12}u^1) - \partial_2(g_{11}u^1))\partial_1 u_2^* \\
& \quad - \mathcal{J}^{-2}(g_{11}\partial_3(g_{12}u^1) + g_{12}\partial_3(g_{11}u^1))\partial_3 u_2^* \Big) \\
\nu_{3,3} = \quad & (\nabla \cdot \mathbf{v}^*)(3)(\partial_2 u^2 + u^2 \mathcal{J}_t/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2} g_{33}(\partial_1(g_{22}u^2) - \partial_2(g_{12}u^2))\partial_1 u_2^* \\
& \quad - \mathcal{J}^{-2}(g_{11}\partial_3(g_{22}u^2) + g_{12}\partial_3(g_{12}u^2))\partial_3 u_2^* \Big) \\
\nu_{3,4} = \quad & (\nabla \cdot \mathbf{v}^*)(3)(\partial_3 u^3 + u^3 \mathcal{J}_\phi/\mathcal{J}) \\
& + \Big(- \mathcal{J}^{-2}(g_{11}\partial_2(g_{33}u^3) + g_{12}\partial_1(g_{33}u^3))\partial_3 u_2^* \Big) \\
\nu_{4,2} = \quad & (\nabla \cdot \mathbf{v}^*)(4)(\partial_1 u^1 + u^1 \mathcal{J}_s/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{11}\partial_3(g_{12}u^1) + g_{12}\partial_3(g_{11}u^1))\partial_2 u_3^* \\
& \quad - \mathcal{J}^{-2}(g_{12}\partial_3(g_{12}u^1) + g_{22}\partial_3(g_{11}u^1))\partial_1 u_3^* \Big) \\
\nu_{4,3} = \quad & (\nabla \cdot \mathbf{v}^*)(4)(\partial_2 u^2 + u^2 \mathcal{J}_t/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{11}\partial_3(g_{22}u^2) + g_{12}\partial_3(g_{12}u^2))\partial_2 u_3^* \\
& \quad - \mathcal{J}^{-2}(g_{12}\partial_3(g_{22}u^2) + g_{22}\partial_3(g_{12}u^2))\partial_1 u_3^* \Big) \\
\nu_{4,4} = \quad & (\nabla \cdot \mathbf{v}^*)(4)(\partial_3 u^3 + u^3 \mathcal{J}_\phi/\mathcal{J}) \\
& + \Big(\mathcal{J}^{-2}(g_{11}\partial_2(g_{33}u^3) + g_{12}\partial_1(g_{33}u^3))\partial_2 u_3^* \\
& \quad - \mathcal{J}^{-2}(g_{12}\partial_2(g_{33}u^3) + g_{22}\partial_1(g_{33}u^3))\partial_1 u_3^* \Big)
\end{aligned}$$

C.4 The continuity equation

$$
\begin{aligned}
P_1' &= \rho^* \rho, \\
Q_{11} &= -\rho^* \Big(u_0^1 \rho_{0s} + u_{0s}^1 \rho_0 + u_0^1 \rho_0 \mathcal{J}_s / \mathcal{J} \\
&\qquad + u_0^2 \rho_{0t} + u_{0t}^2 \rho_0 + u_0^2 \rho_0 \mathcal{J}_t / \mathcal{J} \\
&\qquad + u_0^3 \rho_{0\phi} + u_{0\phi}^3 \rho_0 + u_0^3 \rho_0 \mathcal{J}_\phi / \mathcal{J}) \\
&\quad - D_\perp (g^{11} \rho_s^* \rho_{0s} + g^{12} (\rho_s^* \rho_t + \rho_t^* \rho_s) + g^{22} \rho_t^* \rho_{0t} + g^{33} \rho_\phi^* \rho_{0\phi}) \\
&\quad - (D_\parallel - D_\perp) \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \Big(B_0^1 \rho_{0s} + B_0^2 \rho_{0t} + B_0^3 \rho_{0\phi} \Big) / |B_0|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{11}' &= -\rho^* \Big(u_0^1 \tilde{\rho}_s + u_{0s}^1 \tilde{\rho} + u_0^1 \tilde{\rho} \mathcal{J}_s / \mathcal{J} \\
&\qquad + u_0^2 \tilde{\rho}_t + u_{0t}^2 \tilde{\rho} + u_0^2 \tilde{\rho} \mathcal{J}_t / \mathcal{J} \\
&\qquad + u_0^3 \tilde{\rho}_\phi + u_{0\phi}^3 \tilde{\rho} + u_0^3 \tilde{\rho} \mathcal{J}_\phi / \mathcal{J}) \\
&\qquad - D_\perp (\rho_s^* \tilde{\rho}_s g^{11} + (\rho_s^* \tilde{\rho}_t + \rho_t^* \tilde{\rho}_s) g^{12} + \rho_t^* \tilde{\rho}_t g^{22} + \rho_\phi^* \tilde{\rho}_\phi g^{33}) \\
&\qquad - (D_\parallel - D_\perp) \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \Big(B_0^1 \tilde{\rho}_s + B_0^2 \tilde{\rho}_t + B_0^3 \tilde{\rho}_\phi \Big) / |B_0|^2 \\
Q_{12}' &= -\rho^* (\tilde{u}^1 \rho_{0s} + \tilde{u}_s^1 \rho_0 + \tilde{u}^1 \rho_0 \mathcal{J}_s / \mathcal{J}) \\
Q_{13}' &= -\rho^* (\tilde{u}^2 \rho_{0t} + \tilde{u}_t^2 \rho_0 + \tilde{u}^2 \rho_0 \mathcal{J}_t / \mathcal{J}) \\
Q_{14}' &= -\rho^* (\tilde{u}^3 \rho_{0\phi} + \tilde{u}_\phi^3 \rho_0 + \tilde{u}^3 \rho_0 \mathcal{J}_\phi / \mathcal{J}) \\
Q_{16}' &= -(D_\parallel - D_\perp) \Big\{ \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \frac{1}{\mathcal{J}} \Big(\partial_3 \tilde{A}_1 \rho_{0t} - \partial_2 \tilde{A}_1 \rho_{0\phi} \Big) / |B_0|^2 \\
&\qquad + \Big(B_0^1 \rho_s + B_0^2 \rho_t + B_0^3 \rho_\phi \Big) \frac{1}{\mathcal{J}} \Big(\partial_3 \tilde{A}_1 \rho_t^* - \partial_2 \tilde{A}_1 \rho_\phi^* \Big) / |B_0|^2 \\
&\qquad - |B_0|^{-4} [(\partial_{A_1} |B_0|^2) \tilde{A}_1] \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \Big(B_0^1 \rho_{0s} + B_0^2 \rho_{0t} + B_0^3 \rho_{0\phi} \Big) \Big\}, \\
Q_{17}' &= -(D_\parallel - D_\perp) \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \frac{1}{\mathcal{J}} \Big(\partial_3 \tilde{A}_2 \rho_{0s} - \partial_1 \tilde{A}_2 \rho_{0\phi} \Big) / |B_0|^2 \\
&\qquad + \Big(B_0^1 \rho_s + B_0^2 \rho_t + B_0^3 \rho_\phi \Big) \frac{1}{\mathcal{J}} \Big(\partial_3 \tilde{A}_2 \rho_s^* - \partial_1 \tilde{A}_2 \rho_\phi^* \Big) / |B_0|^2 \\
&\qquad - |B_0|^{-4} [(\partial_{A_2} |B_0|^2) \tilde{A}_2] \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \Big(B_0^1 \rho_{0s} + B_0^2 \rho_{0t} + B_0^3 \rho_{0\phi} \Big) \Big\}, \\
Q_{18}' &= -(D_\parallel - D_\perp) \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \frac{1}{\mathcal{J}} \Big(\partial_2 \tilde{A}_3 \rho_{0s} - \partial_1 \tilde{A}_3 \rho_{0t} \Big) / |B_0|^2 \\
&\qquad + \Big(B_0^1 \rho_s + B_0^2 \rho_t + B_0^3 \rho_\phi \Big) \frac{1}{\mathcal{J}} \Big(\partial_2 \tilde{A}_3 \rho_s^* - \partial_1 \tilde{A}_3 \rho_t^* \Big) / |B_0|^2 \\
&\qquad - |B_0|^{-4} [(\partial_{A_3} |B_0|^2) \tilde{A}_3] \Big(B_0^1 \rho_s^* + B_0^2 \rho_t^* + B_0^3 \rho_\phi^* \Big) \Big(B_0^1 \rho_{0s} + B_0^2 \rho_{0t} + B_0^3 \rho_{0\phi} \Big) \Big\},
\end{aligned}
$$

C.5 The momentum equation

Then for the momentum equation we have the 'radial' component u^1, the right hand side

$$
\begin{aligned}
P_2' &= \rho u_1^* \tilde{u}^1, \\
Q_{22} &= -\rho_0 u_1^* \Big(u_0^1 u_{0s}^1 + u_0^2 u_{0t}^1 + u_0^3 u_{0\phi}^1 \Big) \\
&\quad - \rho_0 u_1^* (u_0^1 u_0^1 \Gamma^1_{11} + 2 u_0^1 u_0^2 \Gamma^1_{12} + u_0^2 u_0^2 \Gamma^1_{22} + u_0^3 u_0^3 \Gamma^1_{33}) \\
&\quad - g^{11} (\rho_{0s} T_0 + \rho T_{os}) - g^{12} (\rho_{0t} T_0 + \rho T_{ot}) \\
&\quad + g^{11} \mathcal{J} (J_0^2 B_0^3 - J_0^3 B_0^2) + g^{12} \mathcal{J} (J_0^3 B_0^1 - J_0^1 B_0^3)
\end{aligned}
$$

and the matrix

$$
\begin{aligned}
Q_{21}' &= -\tilde{\rho} u_1^* (u_0^1 u_{0s}^1 + u_0^2 u_{0t}^1 + u_0^3 u_{0\phi}^1) \\
&\quad - \tilde{\rho} u_1^* (u_0^1 u_0^1 \Gamma^1_{11} + 2 u_0^1 u_0^2 \Gamma^1_{12} + u_0^2 u_0^2 \Gamma^1_{22} + u_0^3 u_0^3 \Gamma^1_{33}) \\
&\quad - u_1^* (g^{11} (\tilde{\rho}_s T_0 + \tilde{\rho} T_{0s}) + g^{12} (\tilde{\rho}_t T_0 + \tilde{\rho} T_{0t})) \\
Q_{22}' &= -\rho_0 u_1^* \Big(u_0^1 \tilde{u}_s^1 + \tilde{u}^1 u_{0s}^1 + u_0^2 \tilde{u}_t^1 + u_0^3 \tilde{u}_\phi^1 + 2 \tilde{u}^1 (u_0^1 \Gamma^1_{11} + u_0^2 \Gamma^1_{12}) \Big) \\
&\quad - \nu(T)\, \nu_{2,2} \\
Q_{23}' &= -\rho_0 u_1^* (u_{0t}^1 \tilde{u}^2 + 2 \tilde{u}^2 (u_0^1 \Gamma^1_{12} + u_0^2 \Gamma^1_{22})) \\
&\quad - \nu(T)\, \nu_{2,3} \\
Q_{24}' &= -\rho_0 u_1^* (u_{0\phi}^1 \tilde{u}^3 + 2 u_0^3 \tilde{u}^3 \Gamma^1_{33}) \\
&\quad - \nu(T)\, \nu_{2,4} \\
Q_{25}' &= -u_1^* (g^{11} (\rho_{0s} \tilde{T} + \rho_0 \tilde{T}_s) + g^{12} (\rho_{0t} \tilde{T} + \rho_0 \tilde{T}_t)) \\
Q_{26}' &= u_1^* (\mathbf{J} \times \mathbf{B})^1 (A_1) \\
Q_{27}' &= u_1^* (\mathbf{J} \times \mathbf{B})^1 (A_2) \\
Q_{28}' &= u_1^* (\mathbf{J} \times \mathbf{B})^1 (A_3)
\end{aligned}
$$

The 'poloidal' component becomes

$$
\begin{aligned}
P'_3 &= \rho u_2^* \tilde{u}^2, \\
Q_{33} &= -\rho_0 u_2^* \Big(u_0^1 u_{0s}^2 + u_0^2 u_{0t}^2 + u_0^3 u_{0\phi}^2 \Big) \\
&\quad - u_2^* (u_0^1 u_0^1 \Gamma_{11}^2 + 2 u_0^1 u_0^2 \Gamma_{12}^2 + u_0^2 u_0^2 \Gamma_{22}^2 + u_0^3 u_0^3 \Gamma_{33}^2) \\
&\quad - g^{12} (\rho_{0s} T_0 + \rho T_{os}) - g^{22} (\rho_{0t} T_0 + \rho T_{ot}) \\
&\quad + g^{12} \mathcal{J} (J_0^2 B_0^3 - J_0^3 B_0^2) + g^{22} \mathcal{J} (J_0^3 B_0^1 - J_0^1 B_0^3)
\end{aligned}
$$

and

$$
\begin{aligned}
Q'_{31} &= -\tilde{\rho} u_2^* (u_0^1 u_{0s}^2 + u_0^2 u_{0t}^2 + u_0^3 u_{0\phi}^2) \\
&\quad - \tilde{\rho} u_2^* (u_0^1 u_0^1 \Gamma_{11}^2 + 2 u_0^1 u_0^2 \Gamma_{12}^2 + u_0^2 u_0^2 \Gamma_{22}^2 + u_0^3 u_0^3 \Gamma_{33}^2) \\
&\quad - u_2^* (g^{12} (\tilde{\rho}_s T_0 + \tilde{\rho} T_{0s}) + g^{22} (\tilde{\rho}_t T_0 + \tilde{\rho} T_{0t})) \\
Q'_{32} &= -\rho_0 u_2^* \Big(u_{0s}^2 \tilde{u}^1 + 2 \tilde{u}^1 (u_0^1 \Gamma_{11}^2 + u_0^2 \Gamma_{12}^2) \Big) \\
&\quad - \nu(T)\, \nu_{3,2} \\
Q'_{33} &= -\rho_0 u_2^* \Big(u_0^1 \tilde{u}_s^2 + u_0^2 \tilde{u}_t^2 + u_{0t}^2 \tilde{u}^2 + u_0^3 \tilde{u}_\phi^2 + 2 \tilde{u}^2 (u_0^1 \Gamma_{12}^2 + u_0^2 \Gamma_{22}^2) \Big) \\
&\quad - \nu(T)\, \nu_{3,3} \\
Q'_{34} &= -\rho_0 u_2^* (\tilde{u}^3 u_{0\phi}^2 + 2 \tilde{u}^3 u_0^3 \Gamma_{33}^2) \\
&\quad - \nu(T)\, \nu_{3,4} \\
Q'_{35} &= -u_2^* (g^{12} (\rho_{0s} \tilde{T} + \rho_0 \tilde{T}_s) + g^{22} (\rho_{0t} \tilde{T} + \rho_0 \tilde{T}_t)) \\
Q'_{36} &= u_2^* (\mathbf{J} \times \mathbf{B})^2 (A_1) \\
Q'_{37} &= u_2^* (\mathbf{J} \times \mathbf{B})^2 (A_2) \\
Q'_{38} &= u_2^* (\mathbf{J} \times \mathbf{B})^2 (A_3)
\end{aligned}
$$

Finally, the toroidal component becomes

$$
\begin{aligned}
P'_4 &= \rho u_{3*} u^3, \\
Q_{44} &= -\rho_0 u_3^* \Big(u_0^1 u_{0s}^3 + u_0^2 u_{0t}^3 + u_0^3 u_{0\phi}^3 \Big) \\
&\quad - \rho_0 u_3^* (2 u_0^1 u_0^3 \Gamma_{13}^3 + 2 u_0^2 u_0^3 \Gamma_{23}^3) \\
&\quad - g^{33} (\rho_{0\phi} T_0 + \rho T_{o\phi}) \\
&\quad + g^{33} \mathcal{J} (J_0^1 B_0^2 - J_0^2 B_0^1)
\end{aligned}
$$

and

$$
\begin{aligned}
Q'_{41} &= -\tilde{\rho} u_3^* (u_0^1 u_{0s}^3 + u_0^2 u_{0t}^3 + u_0^3 u_{0\phi}^3) \\
&\quad - \tilde{\rho} u_3^* (2 u_0^1 u_0^3 \Gamma_{13}^3 + 2 u_0^2 u_0^3 \Gamma_{23}^3) \\
&\quad - u_3^* g^{33} (\tilde{\rho}_\phi T_0 + \tilde{\rho} T_{0\phi}) \\
Q'_{42} &= -\rho_0 u_3^* (\tilde{u}^1 u_{0s}^3 + 2 \tilde{u}^1 u_0^3 \Gamma_{13}^3) \\
&\quad - \nu(T)\, \nu_{4,2} \\
Q'_{43} &= -\rho_0 u_3^* (\tilde{u}^2 u_{0t}^3 + 2 \tilde{u}^2 u_0^3 \Gamma_{23}^3) \\
&\quad - \nu(T)\, \nu_{4,3} \\
Q'_{44} &= -\rho_0 u_3^* (u_0^1 \tilde{u}_s^3 + u_0^2 \tilde{u}_t^3 + u_0^3 \tilde{u}_\phi^3 + \tilde{u}^3 u_{0\phi}^3 + 2 \tilde{u}^3 (u_0^1 \Gamma_{13}^3 + u_0^2 \Gamma_{23}^3)) \\
&\quad - \nu(T)\, \nu_{4,4} \\
Q'_{45} &= -u_3^* g^{33} (\rho_{0\phi} \tilde{T} + \rho_0 \tilde{T}_\phi) \\
Q'_{46} &= u_3^* (\mathbf{J} \times \mathbf{B})^3 (A_1) \\
Q'_{47} &= u_3^* (\mathbf{J} \times \mathbf{B})^3 (A_2) \\
Q'_{48} &= u_3^* (\mathbf{J} \times \mathbf{B})^3 (A_3)
\end{aligned}
$$

C.6 The temperature (pressure) equation

$$
\begin{aligned}
P_5' &= T^*T, \\
Q_{55} &= -T^*(u_0^1 T_{0s} + u_0^2 T_{0t} + u_0^3 T_{0\phi}) \\
&\quad + T^*(\gamma-1)T_0\Big(\frac{1}{JR}(u_0^1(J_sR + JR_s) + u_0^2(J_tR + JR_t) + u_0^3(J_\phi R + JR_\phi)) + u_{0s}^1 + u_{0t}^2 + u_{0\phi}^3\Big) \\
&\quad - K_\perp(g^{11}T_s^*T_{0s} + g^{12}(T_s^*T_{0t} + T_t^*T_{0s}) + g^{22}T_t^*T_{0t} + g^{33}T_\phi^*T_{0\phi}) \\
&\quad - (K_\parallel - K_\perp)\Big(B_0^1T_s^* + B_0^2T*_t + B_0^3T_\phi^*\Big)\Big(B_0^1T_{0s} + B_0^2T_{0t} + B_0^3T_{0\phi}\Big)/|B_0|^2
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{51}' &= 0 \\
Q_{52}' &= -T^*\tilde{u}^1T_{0s} + T^*(\gamma-1)T_0\Big(\frac{1}{JR}\tilde{u}^1(J_sR + JR_s) + \tilde{u}_s^1\Big) \\
Q_{53}' &= -T^*\tilde{u}^2T_{0t} + T^*(\gamma-1)T_0\Big(\frac{1}{JR}\tilde{u}^2(J_tR + JR_t) + \tilde{u}_t^2\Big) \\
Q_{54}' &= -T^*\tilde{u}^3T_{0\phi} + T^*(\gamma-1)T_0\Big(\frac{1}{JR}\tilde{u}^3J_\phi R + \tilde{u}_\phi^3\Big) \\
Q_{55}' &= T^*(u_0^1\tilde{T}_s + u_0^2\tilde{T}_t + u_0^3\tilde{T}_\phi) \\
&\quad + T^*(\gamma-1)\tilde{T}\Big(\frac{1}{J}(u_0^1J_s + u_0^2J_t + u_0^3J_\phi) + u_{0s}^1 + u_{0t}^2 + u_{0\phi}^3\Big) \\
&\quad - K_\perp(T_s^*\tilde{T}_sg^{11} + (T_s^*\tilde{T}_t + T_t^*\tilde{T}_s)g^{12} + T_t^*\tilde{T}_tg^{22} + T_\phi^*\tilde{T}_\phi g^{33}) \\
&\quad - (K_\parallel - K_\perp)\Big(B_0^1T_s^* + B_0^2T_t^* + B_0^3T_\phi^*\Big)\Big(B_0^1\tilde{T}_s + B_0^2\tilde{T}_t + B_0^3\tilde{T}_\phi\Big)/|B_0|^2 \\
Q_{56}' &= -(K_\parallel - K_\perp)\Big(B_0^1T_s^* + B_0^2T_t^* + B_0^3T_\phi^*\Big)\Big(\partial_3\tilde{A}_1T_{0t} - \partial_2\tilde{A}_1T_{0\phi}\Big)/|B_0|^2 \\
Q_{57}' &= -(K_\parallel - K_\perp)\Big(B_0^1T_s^* + B_0^2T_t^* + B_0^3T_\phi^*\Big)\Big(\partial_3\tilde{A}_2T_{0s} - \partial_1\tilde{A}_2T_{0\phi}\Big)/|B_0|^2 \\
Q_{58}' &= -(K_\parallel - K_\perp)\Big(B_0^1T_s^* + B_0^2T_t^* + B_0^3T_\phi^*\Big)\Big(\partial_2\tilde{A}_3T_{0s} - \partial_1\tilde{A}_3T_{0t}\Big)/|B_0|^2
\end{aligned}
$$

C.7 The induction equation

Finally, with 'radial' component

$$
\begin{aligned}
P'_6 &= A^{1*}A_1, \\
Q_{66} &= \mathcal{J}A^{1*}(u_0^2B_0^3 - u_0^3B_0^2) \\
&\quad - \mathcal{J}A^{1*}\Big(g^{33}\eta_\phi B_0^1 - (g^{12}\eta_s + g^{22}\eta_t)B_0^3\Big) \\
&\quad - \eta\frac{1}{\mathcal{J}}\Big((-g_{11}\partial_3(g_{12}\tilde{A}_1) + g_{12}\partial_3(g_{11}\tilde{A}_1))B_0^1 \\
&\qquad\qquad + (-g_{12}\partial_3(g_{12}\tilde{A}_1) + g_{22}\partial_3(g_{11}\tilde{A}_1))B_0^2 \\
&\qquad\qquad + (-g_{33}\partial_2(g_{11}\tilde{A}_1) + g_{33}\partial_1(g_{12}\tilde{A}_1))B_0^3\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
Q'_{61} &= 0 \\
Q'_{62} &= 0 \\
Q'_{63} &= \quad \mathcal{J}A^{1*}\tilde{u}^2B_0^3 \\
Q'_{64} &= -\mathcal{J}A^{1*}\tilde{u}^3B_0^2 \\
Q'_{65} &= 0 \\
Q'_{66} &= A^{1*}(-u_0^2\partial_2\tilde{A}_1 - u_0^3\partial_3\tilde{A}_1) \\
&\quad - A^{1*}\Big(g^{33}\eta_\phi\partial_3\tilde{A}_1 - (g^{12}\eta_s + g^{22}\eta_t)\partial_2\tilde{A}_1\Big) \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((-g_{12}\partial_3(g_{12}A^{1*}) + g_{22}\partial_3(g_{11}A^{1*}))\partial_3\tilde{A}_1 \\
&\qquad\qquad + (g_{33}\partial_2(g_{11}A^{1*}) - g_{33}\partial_1(g_{12}A^{1*}))\partial_2\tilde{A}_1\Big) \\
Q'_{67} &= A^{1*}(u_0^2\partial_1\tilde{A}_2 + (g^{12}\eta_s + g^{22}\eta_t)\partial_1\tilde{A}_2) \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((g_{11}\partial_3(g_{12}A^{1*}) - g_{12}\partial_3(g_{11}A^{1*}))\partial_3\tilde{A}_2 \\
&\qquad\qquad + (-g_{33}\partial_2g_{11}A^{1*} + g_{33}\partial_1g_{12}A^{1*})\Big) \\
Q'_{68} &= A^{1*}(u_0^3\partial_1\tilde{A}_3) + g^{33}(A^{1*}\eta_\phi)\partial_1\tilde{A}_3 \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((-g_{11}\partial_3(g_{12}A^{1*}) + g_{12}\partial_3(g_{11}A^{1*}))\partial_2\tilde{A}_3 \\
&\qquad\qquad + (g_{12}\partial_3(g_{12}A^{1*}) - g_{22}\partial_3(g_{11}A^{1*}))\partial_1\tilde{A}_3\Big)
\end{aligned}
$$

For the 'poloidal' component of the vector potential we have

$$\begin{aligned}
P'_7 &= A^{2*}A_2, \\
Q_{77} &= \mathcal{J}A^{2*}(u_0^3 B_0^1 - u_0^1 B_0^3) \\
&\quad - \mathcal{J}A^{2*}\Big(g^{33}\eta_\phi B_0^1 - (g^{11}\eta_s + g^{12}\eta_t)B_0^3\Big) \\
&\quad - \eta\frac{1}{\mathcal{J}}\Big((-g_{11}\partial_3(g_{22}A^{2*}) + g_{12}\partial_3(g_{12}A^{2*}))B_0^1 \\
&\qquad\qquad + (g_{22}\partial_3(g_{12}A^{2*}) - g_{12}\partial_3(g_{22}A^{2*}))B_0^2 \\
&\qquad\qquad + (g_{33}\partial_1(g_{22}A^{2*}) - g_{33}\partial_2(g_{12}A^{2*}))B_0^3\Big)
\end{aligned}$$

and

$$\begin{aligned}
Q'_{71} &= 0 \\
Q'_{72} &= -\mathcal{J}A^{2*}\tilde{u}^1 B_0^3 \\
Q'_{73} &= 0 \\
Q'_{74} &= \quad \mathcal{J}A^{2*}\tilde{u}^3 B_0^1 \\
Q'_{75} &= 0 \\
Q'_{76} &= A^{2*}(u_0^1 + g^{11}\eta_s + g^{12}\eta_t)\partial_2\tilde{A}_1 \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((-g_{12}\partial_3(g_{22}A^{2*}) + g_{22}\partial_3(g_{12}A^{2*}))\partial_3\tilde{A}_1 \\
&\qquad\qquad + (g_{33}\partial_2(g_{12}A^{2*}) - g_{33}\partial_1(g_{22}A^{2*}))\partial_2\tilde{A}_1\Big) \\
Q'_{77} &= A^{2*}(-(u_0^3 + g^{33}\eta_\phi)\partial_3\tilde{A}_2 - (u_0^1 + (g^{11}\eta_s + g^{12}\eta_t))\partial_1\tilde{A}_2) \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((g_{11}\partial_3(g_{22}A^{2*}) - g_{12}\partial_3(g_{12}A^{2*}))\partial_3\tilde{A}_2 \\
&\qquad\qquad + (g_{33}\partial_1(g_{22}A^{2*}) - g_{33}\partial_2(g_{12}A^{2*}))\partial_1\tilde{A}_2\Big) \\
Q'_{78} &= A^{2*}(u_0^3 + g^{33}\eta_\phi)\partial_2\tilde{A}_3 \\
&\quad - \eta\frac{1}{\mathcal{J}^2}\Big((-g_{11}\partial_3(g_{22}A^{2*}) + g_{12}\partial_3(g_{12}A^{2*}))\partial_2\tilde{A}_3 \\
&\qquad\qquad + (g_{12}\partial_3(g_{22}A^{2*}) - g_{22}\partial_3(g_{12}A^{2*}))\partial_1\tilde{A}_3\Big)
\end{aligned}$$

The toroidal component ('ψ') becomes

$$
\begin{aligned}
P_8' &= A^{3*}A_3,\\
Q_{88} &= \mathcal{J}A^{3*}(u_0^1B_0^2 - u_0^2B_0^1)\\
&\quad - \mathcal{J}A^{3*}\Big((g^{12}\eta_s + g^{22}\eta_t)B_0^1 - (g^{11}\eta_s + g^{12}\eta_t)B_0^2\Big)\\
&\quad - \eta\frac{1}{\mathcal{J}}\Big((g_{11}\partial_2(g_{33}A^{3*}) - g_{12}\partial_1(g_{33}A^{3*}))B_0^1\\
&\qquad\qquad + (g_{12}\partial_2(g_{33}A^{3*}) - g_{22}\partial_1(g_{33}A^{3*}))B_0^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_{81}' &= 0\\
Q_{82}' &= \quad \mathcal{J}A^{3*}\tilde{u}^1B_0^2\\
Q_{83}' &= -\mathcal{J}A^{3*}\tilde{u}^2B_0^1\\
Q_{84}' &= 0\\
Q_{85}' &= 0\\
Q_{86}' &= A^{3*}(u_0^1 + (g^{11}\eta_s + g^{12}\eta_t))\partial_3\tilde{A}_1\\
&\quad + A^{3*}(g^{12}\eta_s + g^{22}\eta_t)\partial_2\tilde{A}_1\\
&\quad - \frac{\eta}{\mathcal{J}^2}\Big(g_{12}\partial_2(g_{33}A^{3*}) - g_{22}\partial_1(g_{33}A^{3*})\Big)\partial_3\tilde{A}_1\\
Q_{87}' &= A^{3*}(u_0^2\partial_3\tilde{A}_2 - (g^{12}\eta_s + g^{22}\eta_t)\partial_1\tilde{A}_2)\\
&\quad - \frac{\eta}{\mathcal{J}^2}\Big(- g_{11}\partial_2(g_{33}A^{3*}) + g_{12}\partial_1(g_{33}A^{3*})\Big)\partial_3\tilde{A}_2\\
Q_{88}' &= A^{3*}((-u_0^1 - (g^{11}\eta_s + g^{12}\eta_t))\partial_1\tilde{A}_3 - u_0^2\partial_2\tilde{A}_3)\\
&\quad - \frac{\eta}{\mathcal{J}^2}\Big((g_{11}\partial_2(g_{33}A^{3*}) - g_{12}\partial_1(g_{33}A^{3*}))\partial_2\tilde{A}_3\\
&\qquad\qquad + (-g_{12}\partial_2(g_{33}A^{3*}) + g_{22}\partial_1(g_{33}A^{3*}))\partial_1\tilde{A}_3\Big)
\end{aligned}
$$

C.8 Definitions

Here, the following abbreviations (definitions) have been used:

$$
\begin{aligned}
B_0^1 &= \frac{1}{\mathcal{J}}(\partial_2 A_{03} - \partial_3 A_{02}),\\
B_0^2 &= \frac{1}{\mathcal{J}}(\partial_3 A_{01} - \partial_1 A_{03}),\\
B_0^3 &= g^{33}F + \frac{1}{\mathcal{J}}(\partial_1 A_{02} - \partial_2 A_{01}),\\
|B_0|^2 &= g_{11}(B_0^1)^2 + 2g_{12}B_0^1 B_0^2 + g_{22}(B_0^2)^2 + g_{33}(B_0^3)^2,
\end{aligned}
$$

and furthermore, as e.g. $s_x = \frac{1}{J}y_t$ with J the two-dimensional Jacobian (see section A)

$$
\begin{aligned}
g^{11} &= \frac{1}{J^2}(x_t^2 + y_t^2), & g_{11} &= x_s^2 + y_s^2,\\
g^{12} &= -\frac{1}{J^2}(x_s x_t + y_s y_t) & g_{12} &= x_s x_t + y_s y_t\\
g^{22} &= \frac{1}{J^2}(x_s^2 + y_s^2), & g_{22} &= x_t^2 + y_t^2,\\
g^{33} &= \frac{1}{R^2}, & g_{33} &= R^2.
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial_1 g^{11} &= \frac{1}{J^2}(2x_t x_{st} + 2y_t y_{st}) - \frac{2}{J^3}(x_{ss}y_t + x_s y_{st} - x_{st}y_s - x_t y_{ss})(x_t^2 + y_t^2),\\
\partial_2 g^{11} &= \frac{1}{J^2}(2x_t x_{tt} + 2y_t y_{tt}) - \frac{2}{J^3}(x_{st}y_t + x_s y_{tt} - x_{tt}y_s - x_t y_{st})(x_t^2 + y_t^2),\\
\partial_1 g^{12} &= -\frac{1}{J^2}(x_{ss}x_t + x_s x_{st} + y_{ss}y_t + y_s y_{st}) - \frac{2}{J^3}(x_{ss}y_t + x_s y_{st} - x_{st}y_s - x_t y_{ss})(x_s x_t + y_s y_t)\\
\partial_2 g^{12} &= -\frac{1}{J^2}(x_{st}x_t + x_s x_{tt} + y_{st}y_t + y_s y_{tt}) - \frac{2}{J^3}(x_{st}y_t + x_s y_{tt} - x_{tt}y_s - x_t y_{st})(x_s x_t + y_s y_t)\\
\partial_1 g^{22} &= \frac{1}{J^2}(2x_s x_{ss} + 2y_s y_{ss}) - \frac{2}{J^3}(x_{ss}y_t + x_s y_{st} - x_{st}y_s - x_t y_{ss})(x_s^2 + y_s^2),\\
\partial_2 g^{22} &= \frac{1}{J^2}(2x_s x_{st} + 2y_s y_{st}) - \frac{2}{J^3}(x_{ss}y_t + x_s y_{tt} - x_{tt}y_s - x_t y_{st})(x_s^2 + y_s^2),\\
\partial_1 g^{33} &= -\frac{2}{R^3}R_s,\\
\partial_2 g^{33} &= -\frac{2}{R^3}R_t,\\
\partial_3 g^{33} &= -\frac{2}{R^3}R_\phi,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_1 g_{11} &= 2x_s x_{ss} + 2y_s y_{ss}, \\
\partial_2 g_{11} &= 2x_s x_{st} + 2y_s y_{st}, \\
\partial_1 g_{12} &= x_{ss} x_t + x_s x_{st} + y_{ss} y_t + y_s y_{st}, \\
\partial_2 g_{12} &= x_{st} x_t + x_s x_{tt} + y_{st} y_t + y_s y_{tt}, \\
\partial_1 g_{22} &= 2x_t x_{st} + 2y_t y_{st}, \\
\partial_2 g_{22} &= 2x_t x_{tt} + 2y_t y_{tt}, \\
\partial_1 g_{33} &= 2RR_s, \\
\partial_2 g_{33} &= 2RR_t, \\
\partial_3 g_{33} &= 2RR_\phi.
\end{aligned}
$$

The Christoffel symbols become

$$
\begin{aligned}
\Gamma^1_{11} &= \frac{J^2}{R^4}\left(g_{22}(x_s x_{ss} + y_s y_{ss}) - g_{12}(x_t x_{ss} + y_t y_{ss}) - R(R_s g_{11} g_{22} + R_t g_{11} g_{12} - 2R_s g_{12} g_{12})\right) \\
\Gamma^1_{12} &= \frac{J^2}{R^4}\left(g_{22}(x_s x_{st} + y_s y_{st}) - g_{12}(x_t x_{st} + y_t y_{st}) - R(R_t g_{11} g_{22} - R_s g_{12} g_{22})\right) \\
\Gamma^1_{22} &= \frac{J^2}{R^4}\left(2g_{22}(x_s x_{tt} + y_s y_{tt}) - g_{12}(x_t x_{tt} + y_t y_{tt}) - R(R_t g_{22} g_{12} - R_s g_{22} g_{22})\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{11} &= \frac{J^2}{R^4}\left(-g_{12}(x_s x_{ss} + y_s y_{ss}) + g_{11}(x_t x_{ss} + y_t y_{ss}) + R(R_t g_{11} g_{11} - R_s g_{11} g_{12})\right) \\
\Gamma^2_{12} &= \frac{J^2}{R^4}\left(-g_{12}(x_s x_{st} + y_s y_{st}) + g_{11}(x_t x_{st} + y_t y_{st}) + R(R_t g_{11} g_{12} - R_s g_{11} g_{22})\right) \\
\Gamma^2_{22} &= \frac{J^2}{R^4}\left(-g_{12}(x_s x_{tt} + y_s y_{tt}) + g_{11}(x_t x_{tt} + y_t y_{tt}) - R(R_s g_{12} g_{22} - 2R_t g_{12} g_{12} + R_t g_{11} g_{22})\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^1_{33} &= \frac{J^2}{R}\left(-g_{22} R_s + g_{12} R_t\right) \\
\Gamma^2_{33} &= \frac{J^2}{R}\left(g_{12} R_s - g_{11} R_t\right) \\
\Gamma^3_{13} &= \frac{R_s}{R}, \\
\Gamma^3_{23} &= \frac{R_t}{R}.
\end{aligned}
$$

Note that in the toroidally axisymmetric equilibria in general $\partial_\phi \equiv 0$.

D The current terms

The current terms in the induction equation are written out for future reference:

$$(\nabla\times\mathbf{A}^*)\cdot(\nabla\times\mathbf{A}) = \frac{1}{\mathcal{J}}\mathbf{a}_k\varepsilon^{ijk}\partial_i g_{jh}A^{*h}\,\frac{1}{\mathcal{J}}\mathbf{a}_k\varepsilon^{ijk}\partial_i A_j,$$

with for $h=1$

$$\begin{aligned}
(\nabla\times g_{i1}A^{1*})\cdot(\nabla\times\mathbf{A}) &= \frac{1}{\mathcal{J}}\Big(\mathbf{a}_1(-\partial_3 g_{12}A^{1*}) + \mathbf{a}_2(\partial_3 g_{11}A^{1*}) + \mathbf{a}_3(\partial_1 g_{12}A^{1*} - \partial_2 g_{11}A^{1*})\Big)\cdot\mathbf{B},\\
&= \frac{1}{\mathcal{J}}\begin{pmatrix} & -\partial_3 g_{12}A^{1*}\\ \partial_3 g_{11}A^{1*} & \\ -\partial_2 g_{11}A^{1*} & +\partial_1 g_{12}A^{1*}\end{pmatrix}\cdot\begin{pmatrix}B^1\\B^2\\B^3\end{pmatrix}\\
&= \frac{1}{\mathcal{J}^2}\begin{pmatrix} & -\partial_3 g_{12}A^{1*}\\ \partial_3 g_{11}A^{1*} & \\ -\partial_2 g_{11}A^{1*} & +\partial_1 g_{12}A^{1*}\end{pmatrix}\cdot\begin{pmatrix} & \partial_2A_3-\partial_3A_2\\ & \partial_3A_1-\partial_1A_3\\ \mathcal{J}g^{33}F+ & \partial_1A_2-\partial_2A_1\end{pmatrix}
\end{aligned}$$

for $h=2$

$$\begin{aligned}
(\nabla\times g_{i2}A^{2*})\cdot(\nabla\times\mathbf{A}) &= \frac{1}{\mathcal{J}}\Big(\mathbf{a}_1(-\partial_3 g_{22}A^{2*}) + \mathbf{a}_2(\partial_3 g_{12}A^{2*}) + \mathbf{a}_3(\partial_1 g_{22}A^{2*} - \partial_2 g_{12}A^{2*})\Big)\cdot\mathbf{B},\\
&= \frac{1}{\mathcal{J}}\begin{pmatrix} -\partial_3 g_{22}A^{2*} & \\ & \partial_3 g_{12}A^{2*}\\ \partial_1 g_{22}A^{2*} & -\partial_2 g_{12}A^{2*}\end{pmatrix}\cdot\begin{pmatrix}B^1\\B^2\\B^3\end{pmatrix}\\
&= \frac{1}{\mathcal{J}^2}\begin{pmatrix} -\partial_3 g_{22}A^{2*} & \\ & \partial_3 g_{12}A^{2*}\\ \partial_1 g_{22}A^{2*} & -\partial_2 g_{12}A^{2*}\end{pmatrix}\cdot\begin{pmatrix} & \partial_2A_3-\partial_3A_2\\ & \partial_3A_1-\partial_1A_3\\ \mathcal{J}g^{33}F+ & \partial_1A_2-\partial_2A_1\end{pmatrix}
\end{aligned}$$

and $h=3$

$$\begin{aligned}
(\nabla\times g_{i3}A^{3*})\cdot(\nabla\times\mathbf{A}) &= \frac{1}{\mathcal{J}}\Big(\mathbf{a}_1(\partial_2 g_{33}A^{3*}) + \mathbf{a}_2(-\partial_1 g_{33}A^{3*})\Big)\cdot\mathbf{B},\\
&= \frac{1}{\mathcal{J}}\begin{pmatrix}\partial_2 g_{33}A^{3*}\\ -\partial_1 g_{33}A^{3*}\\ 0\end{pmatrix}\cdot\begin{pmatrix}B^1\\B^2\\B^3\end{pmatrix}\\
&= \frac{1}{\mathcal{J}^2}\begin{pmatrix}\partial_2 g_{33}A^{3*}\\ -\partial_1 g_{33}A^{3*}\\ 0\end{pmatrix}\cdot\begin{pmatrix} & \partial_2A_3-\partial_3A_2\\ & \partial_3A_1-\partial_1A_3\\ \mathcal{J}g^{33}F+ & \partial_1A_2-\partial_2A_1\end{pmatrix}
\end{aligned}$$

E The reduced MHD (RMHD) equations

The full MHD equations read

$$
\begin{aligned}
\partial_t \rho &= -\nabla \cdot (\rho \mathbf{v}) + \nabla \cdot (D_\perp \nabla_\perp \rho + D_\parallel \nabla_\parallel \rho) + S_\rho, \\
\rho \partial_t \mathbf{v} &= -\rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla(\rho T) + \mathbf{J} \times \mathbf{B} + \mu \nabla^2 \mathbf{v}, \\
\partial_t T &= -\mathbf{v} \cdot \nabla T - (\gamma - 1) T \nabla \cdot \mathbf{v} + \nabla \cdot (K_\perp \nabla_\perp T + K_\parallel \nabla_\parallel T) + S_T, \\
\partial_t \mathbf{A} &= -\eta \mathbf{J} + \mathbf{v} \times \mathbf{B}
\end{aligned}
$$

When we apply the following symplifications:

$$
\begin{aligned}
\mathbf{B} &= R_0 B_0 \mathbf{a}^3 + R_0 \nabla \psi \times \mathbf{a}^3, \\
\mathbf{v} &= -\frac{R^2}{R_0 B_0} \nabla \phi \times \mathbf{a}^3 \\
J &= \Delta^* \psi, \qquad w = \nabla \cdot \nabla_\perp \phi
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\partial_t \rho &= -(1 + \epsilon x)[\phi, \rho] + 2\epsilon \rho \partial_y \phi + \nabla \cdot (D_\perp \nabla_\perp \rho) + S_\rho, \\
\partial_t w &= -(1 + \epsilon x)[\phi, w] + 2\epsilon \partial_y \phi + \frac{1}{1 + \epsilon x}[\psi, J] - \frac{\epsilon}{(1 + \epsilon x)^2} \partial_3 J + \nu \nabla_\perp^2 w, \\
\rho \partial_t T &= -(1 + \epsilon x)\rho[\phi, T] + 2\epsilon \rho T \partial_y \phi + \nabla \cdot (\mathsf{K} \nabla T) + S_T, \\
\partial_t \psi &= -(1 + \epsilon x)[\phi, \psi] + \eta \Delta^* \psi - \epsilon \partial_3 \phi.
\end{aligned}
$$

with variables

$$
\mathbf{y} \equiv [\psi, \phi, w, \rho, T, j]^T
$$